Causal, homogeneous, and multidimensional structures of manifolds and geometries applied in mathematical cosmology

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1. Introduction

Topological and differentiable (real) manifolds are nowadays present in almost all mathematical models of natural phenomena or engineering problems containing some differential equation. Often the manifold under consideration carries an additional structure characteristic for the model, and its intended range of validity. Causal, homogeneous, and multidimensional structures shall be the topic below. They find their natural applications within the range of current mathematical cosmology. The latter today includes not only classical relativistic cosmology but in particular also its extensions towards quantum geometry and to dimensions different than $1 + 3$. Accordingly, already one third of the International Seminar on *Current Topics in Mathematical Cosmology*, 1998 in Potsdam, was dedicated to these modern extensions [1].

It is well known that a geometry given by a differentiable connection on a manifold need not be related to any metric, although vice versa any differentiable metric defines a unique metric compatible connection, namely the Levi-Civita connection.

Similarly, below we define rather general causal structures on (real) manifolds which do not imply the existence of a metric. Even a non-differentiable topological manifold may carry a causal structure. However, in this case there is no definition neither of a connection 1-form, nor of a curvature tensor, nor of a metric. Even if the manifold is differentiable, a differentiable causal structure does not imply the existence of a (conformal) metric (of any signature). Moreover, given a causal structure and a metric of corresponding signature, both need not necessarily be compatible with each other.

Simple, simply laced singularities (often also known as ADE singularities according to the Cartan type of corresponding Lie algebras) provide a local model for a rather general class of structures on manifolds which might also be called pseudo-causal. Relativity and cosmology usually refer to the conical singularity as local model for causality. Therefore, unless stated otherwise, here we will refer to causal structures which are given as local cone structures.

A Hausdorff (i.e. T_2-strongly separating) topological manifold M without boundary is topologically homogeneous per definition, since the neighborhoods of all points are modeled over the same standard vector space. Note however that a homogeneous C^r-structure on a closed manifold need not be C^{r+1}-homogeneous, while the vice versa is always true.

The homogeneity of a manifold M can be expressed by the existence of a transitive group action on the manifold. A homogeneous manifold is characterized locally by its homeomorphism group. If M is homogeneous, any two points are connected by a local homeomorphism, and the (local) homeomorphism group $\mathrm{Hom}M$ acts on the entire manifold. (Note: Here we do not consider global homeomorphisms which are a topic in its own.) If M contains boundary points, one can always achieve topological homogeneity by restricting the manifold to its interior.

However, if the manifold carries more structure, homeomorphisms have to preserve not only the local topology but also the additional structure. So, a C^r-differential structure on a manifold restricts the homeomorphism to C^r-diffeomorphisms. Similarly, a causal structure on a manifold restricts its homeomorphisms to those which preserve the causal structure. Likewise, a diffeomorphism of a Riemannian manifold (M, g) is a homeomorphism of the latter, iff it is an isometry.

In general with any given structure on a manifold, the structure-preserving homeomorphisms form a subgroup of the homeomorphism group. A structure s on a manifold M is homogeneous, iff the structure preserving homeomorphisms group $\mathrm{Hom}(M, s)$ acts transitively on M.

The cosmological application of causal topological manifolds refers to the requirement of quantum geometric gravity to implement the local topological content of general relativity in absence of a background geometry. Homogeneous and multidimensional geometries both relate to symmetries present either in the large scale cosmological structure or due to unification of general relativity with other fundamental symmetries in higher dimension.

Dynamically the different symmetries reflect themselves in fundamental forces: gravity resulting from general relativity, and other forces from other symmetries.

Rather than trying to give an exhaustive discussion of causal, homogeneous, and multidimensional structures in mathematical cosmology, below the focus will be on some selected aspects of these structures which are of particular relevance in current research. Topologically defined causal structures are of importance when the geometry itself is subject to quantization, i.e. for canonical quantum gravity in particular. Homogeneity structures are not only suggested by phenomenological symmetries in cosmology, but the construction of a classifying space of local homogeneous geometries in terms of scalar algebraic invariants, and the relation to a similar classifying space of the isometries gives systematic insight about the stability of a particular isometry under deformation of the geometry. Multidimensional and at least partially homogeneous geometries are essential for the existence of elegant effective mathematical models, like the effective sigma-model of multidimensional geometry, which result in clarification, generalization, and prediction of structures and results inherent in the geometrical content of modern unification approaches such as M-theory, and also may be sufficiently tractable in order to be a basis for further investigations on higher dimensional geometry, like their canonical quantization via midisuperspaces, or the imprint of features of the higher-dimensional geometry in $3+1$ dimensions ("windows towards extra-dimensions"), e.g. in the Hawking temperature of black holes extended by p-branes. Classical general relativity includes traditionally classification and analysis of solutions to the field equations of Einstein-Hilbert actions with various matter terms on 4-dimensional Lorentzian manifolds within some C^r-category of differentiability, where most results are given for $r \geq 2$ and often C^∞-smoothness is assumed. Traditional mathematical cosmology is targeted to investigate those solutions, which model a hypothetical large scale structure of our $3 + 1$-dimensional universe under more or less realistic assumptions about its matter content. Roughly those models may be divided in those which are spatially homogeneous, and those which are spatially in homogeneous. With an appropriate choice of time coordinates, homogeneous spatial 3-manifolds result here as hypersurfaces of constant time.

In traditional classical general relativity and cosmology, the causal structure is given a priori by the Lorentzian signature of the geometry. Continuous efforts towards a consistent canonical quantization of the geometry underlying general relativity, have risen new issues on the possible treatment of causality. Canonical quantization has first been executed for cosmological models with effective minisuperspaces of homogeneous spatial geometries. Investigation of such models, and the discussion of the state and relation of general relativity and quantum mechanics has emerged into to field of quantum cosmology. Among the issues at dispute in quantum cosmology form its early days was the question of the role of time, and the concept of causality. That issues prevailed in the more general recent approaches towards quantization of general relativity. The axiomatic setting of algebraic quantum field theory (AQFT) is most appropriate to discuss the conceptual issue of causality in any general relativistic quantum field theory (QFT) and in any theory of quantum geometry arising as a quantization of general relativity. A quantum geometry may result from a canonical quantization scheme of classical geometry, on a fixed topology of the underlying manifold. A causal structure is, locally and globally, in general stronger than the standard topological or differentiable structure of manifolds, but it may be much weaker than a metric structure, and even may be weaker than a conformal metric structure. Hence the question now is: Should the causal structure be treated under quantization as a background, or should it be treated dynamically like the geometry itself ?

Under a direct generally covariant quantization method a quantum concept of causality should result dynamically together with quantum geometry. Hence like the geometry itself, also causality would become fuzzy. However the approach in canonical quantization is different. A foliation of the underlying (space-time) manifold is usually assumed, and under the pretext that changes of the foliation are gauge transformations for quantum geometries, the geometry of each slice is quantized first canonically, and then weaved through the whole manifold. With a representation of quantum 3-geometry by spin networks, the quantized 4-geometry is given in terms of spin foams, which may be viewed as weaves of spin networks on slices through the whole manifold. For that approach it is crucial to have a concept of causality, i.e. to know whether an edge of the spin foam is space-like (i.e. it belongs to a spin network on a slice), or time-like, or null. For this purpose it is useful to have a notion of causality which is purely topological, independent of the metric and any part of the geometry which is subject to quantization.

The analysis of $3+1$-dimensional solutions with singularities, and more particularly of black hole solutions, are by now an integral part of mathematical cosmology. The static, spherically symmetric solutions like the well known Schwarzschild solution are spatially inhomogeneous, but still hypersurface homogeneous 4-geometries. The homogeneous hypersurfaces here are hypersurfaces of constant radial parameter.

The physical idea of unification of gravity with additional structures related to other forces, starting from the Kaluza-Klein theory up to present days M-theory, had its impact on the development of mathematical cosmology too. The unification idea topologically reflects in a fiber bundle structure with the $3+1$-dimensional space-time manifold underlying the gravitational field as base manifold and an additional manifold of carrying other fields as fiber. Subsequently the dimensionality of the total manifold in mathematical models of cosmology became $1+3+d$, with extra dimension d from the fiber manifold. Topologically it is then an immediate generalization to consider multidimensional manifolds consisting of such a fiber bundle where the fiber admits a multidimensional decomposition as a direct product of a finite number n of factor spaces of dimensions $d_1, \ldots, d_n$. However, the physical unification idea is implemented not only within the topological category of manifolds, but more particularly also on the level of (pseudo-)Riemannian geometries on multidimensional manifolds. The physical idea is to extract the $3+1$-dimensional space-time with additional fields thereon, as a solution of the dimensional reduction of the Einstein-Hilbert action (possibly with few extra terms which may contain few elementary fields) on a multidimensional manifold to the external $3+1$-dimensional manifold. The dimensional reduction requires firstly that the geometry admits a multidimensional decomposition (as tensor on) on the multidimensional manifold, and secondly that all internal factor spaces be homogeneous (and possible elementary fields be homogeneous on the internal manifold fiber).

By now multidimensional cosmological models have been investigated quite detailed both, for classes of classical spatially homogeneous solutions (e.g. solutions related to Toda systems or solutions with spherical symmetry) and for minisuperspace models with a finite-dimensional phase space admitting canonical quantization, and elementary solutions of the quantized scalar constraint, the Wheeler-deWitt equation, which under appropriate superposition yield solution of quantum cosmology with defined boundary conditions.

The discovery of the multidimensional σ-model extended these results in two directions. Firstly, unlike the first multidimensional cosmological models, the multidimensional σ-model admits multidimensional geometries which are not spatially homogeneous, and which may have a base manifold of arbitrary dimension. Secondly, the multidimensional σ-model turned out to be the right structure to describe the geometric content of p-branes, charged membrane like structures in M-theory (the contemporary generalization of string theory), classically, in arbitrarily curved backgrounds, just on the basis of multidimensional general relativity and antisymmetric fields generalizing the Maxwell field, i.e. within the setting of manifolds and bosonic fields, without the need to utilize algebraic concepts like dualities or supersymmetry.

The work below is divided in three large parts. The first part is dealing with causal structures and applications to AQFT and quantum gravity based conceptually on the work of [2], [3], [4], [5], [6], [7], [8]. Its more specific consequences for quantum black holes and strings [9], relate also to some earlier work [10], [11], [12] on tubular networks in space-time.

The second part is devoted to homogeneous structures, in particular to classifying spaces of homogeneous geometries and their isometries. It is based on [13], [14], [15]. It is related to applications in multidimensional cosmology in [16], and to earlier work of [17], [18], [19], [20], [21], [22].

The third part is on multidimensional geometry and its effective σ-model description. It is based on work described in [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], and [33]. The topic is also related to other work within the multidimensional setting [34], [35], [36], and to early investigations on conformal transformations in [37], [38], and [39]. Some related work on higher-order gravity is discussed in [40]. The general topic of σ-models has been treated by the author already more than a decade ago in [41], when investigating representations of supersymmetries in target spaces appearing in models of statistical physics as a more particular aspect.

The work is organized by the following sections.

Sect. 2 deals with topological definitions of causal structures of different strengths on topological or differentiable manifolds. The existence of a topological causal structure is independent of the existence of a metric structure. In [2] topological causal structures have been introduced via generalized cones.

After an introduction to local cone structures in Sect. 2.1, Sect. 2.2 deals with causality on topological manifolds, and Sect. 2.3 discusses causal index sets and diffeomorphisms.

Sect. 3 is the first section dedicated to cosmological applications. It deals with applications of the causal structures from the previous section in quantum general relativity.

Sect. 3.1 gives an axiomatic introduction to algebraic QFT on manifolds, generalizing the elegant Haag-Kastler formulation of quantum field theory on Minkowski space. This approach is particularly useful in order to clarify the general structure of the theory. The most basic ingredient is a causal net of local *-algebras on a net of localization domains. Investigations in [5] and [3]) searching for a background independent extension of the Haag-Kastler framework to more general causal nets of *-algebras on differentiable manifolds, have shown that certain key properties (isotony, covariance, causality, etc.) should be maintained as far as possible.

In Sect. 3.1.1 we deal with the general axioms of QFT on a differentiable manifold (including in particular isotony and covariance) which do not require a particular notion of causality. Sect. 3.1.2 then suggests axioms for QFT on a manifold with cone causality.

As a much more particular application Sect. 3.2 then deals with quantum geometry as an algebraic QFT. In [4] it was shown how quantum geometry may fit into the framework of algebraic QFT. This setting naturally accommodates spin networks on graphs. This was used in [9] to investigate the classical limit of quantum geometry. If quantum gravity is a true AQFT with infinitely many vertices, the classical limit yields a tubular network of resolutions of vertices and edges. Remarkably, resolutions like these had already previously been discussed in [10]) from a different point of view.

Sect. 4 is dedicated to homogeneity of different structures. Sect. 4.1 introduces homogeneous manifolds for structures of various strengths Sect. 4.2 focuses on local homogeneous geometries, and more particularly on topological classifying spaces for geometries and corresponding algebras. Sect. 4.2.1 reviews the classifying space of local isometries in fixed real dimension. It is given as non-separating T0-space of corresponding Lie-algebras. In [19] such classifying spaces have been constructed up to real dimension 4. Sect. 4.2.2 identifies the topology of such a classifying space as the dual of the Zariski topology. This can been seen best in the view of Lie algebra cohomology. Sect. 4.2.3 describes the construction classifying spaces of local homogeneous geometries. In particular, a complete classification of local homogeneous Riemannian 3-geometries in terms of scalar geometric invariants is given as in [14]. Given the principal anisotropies, the invariants can be calculated directly from a certain normal form of the Lie-algebra. Following [13] it it is shown how the classifying space of local homogeneous 3-geometries projects to the classifying space of Lie-algebras.

The analogous classification of Lorentzian geometries is still under investigation. One of the main problems here is to control various possible relative orientations of the light cone axis with respect to the principal axes of anisotropy in tangent space.

Sect. 5 is the second section dedicated to cosmological applications. It sketches briefly possible applications of the classifying spaces of algebras and geometries in the context of rigidity of algebraic structures, particularly of isometries. Previously obtained classifying spaces are shown to shed some light on the rigidity of the local isometry type under the variation of Riemannian or Lorentzian geometries.

The complete parametrization of local 3-geometries of a definite class like the homogeneous one is of particular interest for a systematic approach to their canonical quantization. The spatially homogeneous class is of primary importance for quantum cosmology.

Sect. 6 is devoted to multidimensional geometry. In particular the effective sigma-model for multidimensional (Einstein-Hilbert) geometry is introduced as in [24] and its general structure is clarified. The bosonic content of string theory contained in Einstein-Hilbert gravity coupled to p-branes and scalars can be described as an effective sigma-model in lower (say 3+1) dimensions coupled to additional interacting dilatonic fields, provided the higher-dimensional curved (!) spacetime has some (multidimensional) symmetry. Sect. 6.1 reviews the effective σ-model for pure multidimensional geometry as it was first introduced in [26]. Sect. 6.2 extends the σ-model by extra scalars and p+2-forms, following the work of [31]. The sigma-model of [23] provides a systematic geometrical description of bosonic string theory sectors on curved (!) background. Historically in gr-qc/9608020 the first such description has been obtained for a multidimensional spacetime background. Sect. 6.3 presents the general target space structure of the effective sigma-model of [24]. Sect. 6.4 introduces special, particularly useful coordinate gauges on M_0, namely the so-called proper coordinates in 6.4.1 and the so-called harmonic coordinates in 6.4.2 .

It follows Sect. 7 as third and most extensive section dedicated to cosmological applications. In this section we present some of the numerous solutions to the multidimensional sigma-model [24]. Following [32], Sect. 7.1 describes the solutions with Abelian target-space, and Sect. 7.2 is devoted to the general structure of orthobrane solutions with ${}^{(E)}V = 0$. Sect. 7.3 gives the general setting for spherically symmetric p-branes. Solutions with p-branes have been obtained first in [31]. In [23] it was shown that such solutions exist, if the target-space is locally symmetric. Sect. 7.4 describes black holes with electromagnetic (EM) branes. As in [27], we consider both, solutions with orthobranes (in 7.4.1) and solutions with degenerate brane charges $Q_e^2 = Q_m^2$ (in 7.4.2). Generalized static black holes solutions with intersecting p-branes predict interesting physical effects. E.g. the Hawking temperature depends critically on the dimension of the p-brane intersection.

Sect. 7.5 deals with spatially homogeneous solutions as in [30]. After introducing multi-component perfect fluid cosmology in Sect. 7.5.1 and the general multidimensional dynamics for the spatially homogeneous case in Sect. 7.5.2, Sect. 7.5.3 presents the classical solutions of the integrable 3-component model, Sect. 7.5.4 deals with classical wormholes, Sect. 7.5.5 shows how to reconstruct classical potentials. and Sect. 7.5.6

treats solutions for the canonically quantized model.

Following [25] Sect. 7.6 demonstrates the particular role of the Einstein frame for cosmology. Sect. 7.6.1 shows how to obtain solutions in the Einstein frame in general, Sect. 7.6.2 then presents solutions in their original form, and Sect. 7.6.3 gives then corresponding solutions in the Einstein frame.

Sect. 7.7 is devoted to solutions of multidimensional m-component cosmology as described in [28]. In particular, 7.7.1 presents the A_m Toda chain solution, and 7.7.2 gives an example in Einstein frame.

Sect. 7.8 considers the exceptional 2-dimensional case as considered in [26] and [29] corresponding to dilaton gravity. In particular 7.8.1 deals with the reduction from (spatially) inhomogeneous cosmology, 7.8.2 derives 2-dimensional dilaton gravity from 5-d Einstein gravity, 7.8.3 considers a spherically symmetric model coupled to matter, and 7.8.4 presents static solutions and discusses the horizon problem in this setting.

Finally, Sect. 8 discusses the results and gives an outlook to current and future research.

Although the author has tried to eliminate obvious misprints, mistakes, inconsistencies, and to unclear formulations as much it was as possible within the scope of time available, the present dissertation does not claim to be a polished presentation of a closed topic with clinical perfection in detail. The reader should keep in mind that the main goal of the following sections is to give an impression, insight and overview of three main directions of research in mathematical cosmology followed by the author during the recent years.

2. Causal structures

While classical general relativity usually employs a Lorentzian space-time metric, all genuine approaches to quantum gravity are free of such a metric background. This poses the question whether there still exists a notion of structure which captures some essential features of light cones and their mutual relations in manifolds in a purely topological manner without a priori recursion to a Lorentzian metric or a conformal class of such metrics. Below we will see that the answer is positive.

It is a well known folk theorem that the causal structure on a Lorentzian manifold determines its metric up to conformal transformations. In [42] and [43] a path topology for strongly causal space-times was defined which then determined their differential, causal, and conformal structure. In [44] it was shown that the conformal class of a Lorentzian metric can be *re*constructed from the characteristic surfaces of the manifold. Similarly [45] gives a nice proof that the null cones determine the Lorentzian metric (modulo global sign) up to a conformal factor. All these previous results already indicate that the notion of a causal structure could exist indeed in a different and possibly more general setting than that of Lorentzian space-times. However all the previously mentioned investigations in the literature assume a priori the existence of some undetermined Lorentzian metric and then show that it can be determined modulo conformal transformation uniquely by some other structure.

Motivated by the requirements on suitable structures for a theory of quantum gravity, in this paper new notions of causal structure are developed which do not assume a priori existence of any (Lorentzian) metric or conformal metric but rather work on arbitrary topological and differential manifolds.

In Sect. 2.1 weak ($\mathscr{C}$) and strong ($\mathscr{C}^m$) local cone (LC) structures are defined on any topological (or differentiable) manifold M. These structures are given by continuous (or differentiable) families of pointwise homeomorphisms from the standard null cone variety in $\mathbb{R}^{d+1}$ or a manifold thickening thereof respectively into M. In the differentiable case it turns out that a strong LC structure implies the existence of a consistent, conformal Lorentzian metric, while a weak LC structure already implies its uniqueness should one exist. A metric consistent with a strong LC structure has to contain pointwise information about the asymptotic structure of the cone at the vertex. Within a given manifold thickening of the cone at a given point of M, in any neighborhood of the vertex, the cone itself need a priori not at all be related to the null structure spanned by the null geodesics of the metric in this neighborhood. However, if in any point of some region the null geodesics of a metric and the cone of the local cone structure are consistent with each other, this yields consistency of the notions of causality defined by the cone structure and the metric respectively.

Sect. 2.2 gives definitions for causality of increasing strength, each definition being essentially a consistency condition for cones at different points in any open region of M. Cone (C-) causality allows first of all the definition of a causal complement with reasonable properties. It enables us also to define in a topological (differentiable) manner spacelike, null, and timelike curves. We discuss C-causality also in the particular context of a fibration. Generalizations of the most common causality notions for space-times in purely topological terms are provided. In the case of Lorentzian manifolds these notions agree with the usual ones and they assume their usual hierarchy. Finally, precausality is defined as a notion which makes the future and the past of any cone homeomorphic to the future and the past of the standard cone $\mathscr{C}$ in $\mathbb{R}^{d+1}$ respectively.

Sect. 2.3 is introduces causal index sets and causal diffeomorphisms. It also addresses the issue of consistency of a foliation of the manifold with its cone structure and causality, as it arises below in cosmological applications.

Here and below a CAT manifold refers to a Hausdorff (T_2) space with CAT structure, where CAT$=\mathcal{C}^0$ (the topological category) or CAT$\subset\mathcal{C}^1$ (any differentiable category). If CAT$\subset\mathcal{C}^1$, a CAT homeomorphism is a diffeomorphism and a CAT continuous map is a differentiable map. For differentiable categories we also define CAT$_{-1}:=\mathcal{C}^r$ if CAT$=\mathcal{C}^{r+1}$, CAT$_{-1}:=\mathcal{C}^\infty$ if CAT$=\mathcal{C}^\infty$, and CAT$_{-1}:=\mathcal{C}^\omega$ if CAT$=\mathcal{C}^\omega$.

2.1 Local cone structures

In this section we derive local notions of a cone structure on a topological $d+1$-dimensional manifold M (CAT$\subset\mathcal{C}^0$). Let

$$\mathscr{C}:=\{x\in\mathbb{R}^{d+1}:x_0^2=(x-x_0e_0)^2\},\mathscr{C}^+:=\{x\in\mathscr{C}:x_0\geq 0\},\mathscr{C}^-:=\{x\in\mathscr{C}:x_0\leq 0\} \tag{2.1}$$

be the standard (unbounded double) light cone, and the forward and backward subcones in $\mathbb{R}^{d+1}$, respectively.

The standard open interior and exterior of $\mathscr{C}$ is defined as

$$\mathscr{T} := \{x \in \mathbb{R}^{d+1} : x_0^2 > (x - x_0 e_0)^2\}, \mathscr{E} := \{x \in \mathbb{R}^{d+1} : x_0^2 < (x - x_0 e_0)^2\}. \tag{2.2}$$

A *manifold thickening* with thickness $m > 0$ is given as

$$\mathscr{C}^m := \{x \in \mathbb{R}^{d+1} : |x_0^2 - (x - x_0 e_0)^2| < m^2\}, \tag{2.3}$$

The characteristic topological data of the standard cone is encoded in the topological relations of all its manifold subspaces (including in particular its singular vertex O) and among each other.

Typical (CAT) manifold subspaces of $\mathscr{C}$ are the standard future and past cones $\mathscr{C}^{\pm}$, and the standard light rays

$$l(n) := \{x \in \mathscr{C} : x_0 = (x, n)\}, \tag{2.4}$$

where $n \in S^{d-1} \subset p$ is a normal direction in the d-dimensional hyperplane $p := \{x \in \mathbb{R}^{d+1} : (x, y) = 0 \ \forall y \in a\}$ perpendicular to the cone axis $a := \{x \in \mathbb{R}^{d+1} : x = \lambda e_0, \lambda \in \mathbb{R}\}$.

The topological relations between all the CAT manifold subspaces of the cone are the natural data which will be required to be conserved under a homeomorphism of the cone as a topological space into the manifold M at any point p.

Let τ denote the closed sets of the manifold topology of $\mathscr{C} - O$. The set $\mathscr{C}$ can either inherit the induced topology τ_1 from $\mathbb{R}^{d+1}$, which is Hausdorff (places in the original publication [2] which state the contrary are mistaken) but not locally Euclidean, or it can be equipped with a coarser subtopology defined in terms of closed sets as $\tau_2 := \{\{0\} \cup V : V \in \tau\} \bigcup \{V \in \tau\}$, which is locally Euclidean. However τ_2 places geometrically unnatural restrictions on possible submanifolds of $\mathscr{C}$. Hence, unless specified otherwise, $\mathscr{C}$ will be equipped with τ_1.

Definition 1: Let M be a CAT manifold. A (CAT) *(null) cone* at $p \in \mathrm{i}ntM$ is the image $\mathscr{C}_p := \phi_p \mathscr{C}$ of a homeomorphism of topological spaces $\phi_p : \mathscr{C} \to \mathscr{C}_p \subset M$ with $\phi_p(0) = p$, such that
(i) every (CAT) submanifold $N \subset \mathscr{C}$ is mapped (CAT) homeomorphically on a submanifold $\phi_p(N) \subset M$,
(ii) for any two submanifolds $N_1, N_2 \subset \mathscr{C}$ there exist homeomorphisms $\phi_p(N_1) \cap \phi_p(N_2) \cong N_1 \cap N_2$ and $\phi_p(N_1) \cup \phi_p(N_2) \cong N_1 \cup N_2$ of (CAT) manifolds if these are (CAT) manifolds and of topological spaces otherwise, and
(iii) if CAT$\subset \mathcal{C}^1$ then for any two CAT curves $c_1, c_2 :]-\epsilon, \epsilon[\to \mathscr{C}$ with $c_1(0) = c_2(0) = p$ it holds $T_0 c_1 = T_0 c_2 \Leftrightarrow T_p(\phi_p \circ c_1)|_{]-\epsilon,\epsilon[} = T_p(\phi_p \circ c_2)|_{]-\epsilon,\epsilon[}$.

Condition (iii) says that in the differentiable case the well defined notion of transversality of intersections at the vertex is preserved by ϕ_p.

On each homeomorphic cone $\mathscr{C}_p$ at any $p \in \mathrm{i}ntM$, the topology τ_1 or τ_2 of $\mathscr{C}$ yields under ϕ_p likewise a locally non-Euclidean topology $\phi_p(\tau_1)$ or a locally Euclidean one $\phi_p(\tau_2)$.

However, $\phi_p(\tau_2)$ would unnaturally restrict the possible submanifolds of $\mathscr{C}$, while $\phi_p(\tau_1)$ is consistent with the topology induced from M.

Definition 2: An *(ultraweak) cone structure* on M is an assignment $intM \ni p \mapsto \mathscr{C}_p$ of a cone at every $p \in intM$.

A cone structure on M can in general be rather wild, with cones at different points totally unrelated to each other, unless we impose a topological connection between the cones at different points. Most naturally the connection is provided by a notion of continuity of a family $\{\mathscr{C}_p\}_{p\in c}$ of cones $\mathscr{C}_p$ with vertex on a continuous curve c within a region $U \subset M$. Let us define continuity of this family with respect to the topology on $\{\mathscr{C}_p\}_{p\in U\subset M}$ defined by the convergence $\mathscr{C}_p \to \mathscr{C}_{p_0}$ for $p \to p_0$ in all points $p_0 \in U$, where $\mathscr{C}_p \to \mathscr{C}_{p_0}$ iff for each manifold $N_0 \subset \mathscr{C}_{p_0}$ there exists a family of manifolds $N_p \subset \mathscr{C}_p$ such that $N \to N_0$ in the topology induced from M.

This allows to define a local cone (LC) structures.

Definition 3: Let M be a CAT manifold. A *weak ($\mathscr{C}$) local cone (LC) structure* on M is a cone structure which is (CAT) continuous i.e. $\{p \mapsto \mathscr{C}_p\}$ is a (CAT) continuous family.

Given a cone structure one wants to know first of all under which conditions, for given $p \in intM$ an exterior and interior of the cone can be distinguished *locally*, i.e. within $(M - \mathscr{C}_p) \cap U$ for any given open neighborhood $U \ni p$. **Proposition 1:** Let $\forall p \in intM$ exist open (CAT) submanifolds $\mathscr{T}_p$ and $\mathscr{E}_p$ such that the interior of M can be written as the disjoint union $intM = \mathscr{C}_p \mathring{\cup} \mathscr{T}_p \mathring{\cup} \mathscr{E}_p$.
(i) Then $\mathscr{T}_p$ and $\mathscr{E}_p$ can be topologically distinguished locally in any neighborhood of the vertex p if and only if for any neighborhood $U \ni p$ it holds $(\mathscr{T}_p|_U) \ncong (\mathscr{E}_p|_U)$, i.e. $\mathscr{T}_p|_U$ and $\mathscr{E}_p|_U$ are inequivalent (in CAT).
(ii) Given any neighborhood $U \ni p$ assume $\exists\ k \in \mathbb{N}_0 : \Pi_k(\mathscr{T}_p|_U) \neq \Pi_k(\mathscr{E}_p|_U)$, where Π_k denotes the k-homotopy group. Then $\mathscr{T}_p$ and $\mathscr{E}_p$ can be topologically distinguished locally in any neighborhood of the vertex p.
Proof: (i) follows from $U - \mathscr{C}_p|_U = \mathscr{T}_p|_U \mathring{\cup} \mathscr{E}_p|_U$. (ii) holds because homotopy groups are topological invariants. □

It is possible to extend the homeomorphism ϕ_p from $\mathscr{C}$ itself to larger open sets of $\mathbb{R}^{d+1}$. Note however that, although $\mathscr{C}_p = \phi_p(\mathscr{C})$, $\mathscr{T}$ and $\mathscr{E}$ need not be homeomorphic to $\phi_p(\mathscr{T})$ and $\phi_p(\mathscr{E})$ respectively. A notion of precausality is set up below to ensure $\mathscr{E}_p \cong \phi_p(\mathscr{E})$.

A weak LC structure at each point $p \in intM$ defines a characteristic topological space $\mathscr{C}_p$ which is locally Euclidean of codimension 1 everywhere but at p. In particular $\mathscr{C}_p$ does not contain any open $U \ni p$ from the manifold topology of M. However stronger structures can be defined as follows.

Definition 4: Let M be a CAT manifold. A (CAT) *(manifold) thickened cone* of thickness $m > 0$ at $p \in intM$ is the (CAT) image $\mathscr{C}_p^m := \phi_p \mathscr{C}^m$ of a (CAT) homeomorphism ϕ_p of the manifold $\mathscr{C}^m \supset \mathscr{C}$ into M with $\phi_p(0) = p$.

Note that ϕ_p maps $\mathscr{C}$ into $\mathscr{C}_p \subset \mathscr{C}_p^m$. Due to the manifold property the notion of a thickened cone is much simpler than that of a cone itself. It also clear that now the only consistent topology on $\mathscr{C} \subset \mathscr{C}^m$ is τ_1 and correspondingly $\phi_p(\tau_1)$ on $\mathscr{C}_p \subset \mathscr{C}_p^m$.

Definition 5: A *thickened cone structure* on M is an assignment $\mathrm{i}ntM \ni p \mapsto \mathscr{C}_p^{m(p)}$ of a thickened cone at every $p \in \mathrm{i}ntM$.

Note that in general the thickness m can vary from point to point in M Here $m : M \to \mathbb{R}_+$ is an a priori not necessarily continuous function. However an important case even more special than the continuous one is that of constant m.

Definition 6: A *homogeneously thickened cone structure* on M is a thickened cone structure $\mathrm{i}ntM \ni p \mapsto \mathscr{C}_p^m$ with constant thickness m.

Homogeneity of the thickness of a cone structure imposes some regularity on the latter. In general, at least continuity of structures on M is a natural assumption in the topological category.

Definition 7: Let M be a CAT manifold. A *strong ($\mathscr{C}^m$) LC structure* on M is a (CAT) continuous family of (CAT) homeomorphisms $\phi_p : \mathscr{C}^m \to \mathscr{C}_p^{m(p)} \subset M$ with $\phi_p(0) = p$ and such that the thickness m is a CAT function on M.

In particular the conditions of (ii) in Proposition 1 apply for all manifolds of dimension $d+1 > 2$ with a strong LC structure, while a weak LC structure at $p \in \mathrm{i}ntM$ may not be able to distinguish $\mathscr{T}_p|_U$ and $\mathscr{E}_p|_U$ within any $U \ni p$.

Theorem 1: Let M carry a strong LC structure. At any $p \in \mathrm{i}ntM$ there exists an open $U \ni p$ such that:
For $d := \dim M - 1 > 0$ it is $|\Pi_0(\mathscr{T}_p|_U)| = 2$ and $\Pi_{d-1}(\mathscr{E}_p|_U) = \Pi_{d-1}(S^{d-1})$,
for $d > 1$ it is $\Pi_{d-1}(\mathscr{T}_p|_U) = 0$ and $|\Pi_0(\mathscr{E}_p|_U)| = 1$,
for $d = 1$ it is $\Pi_{d-1}(\mathscr{T}_p|_U) = \Pi_{d-1}(\mathscr{E}_p|_U) = \Pi_0(S^0)$, i.e. $|\Pi_0(\mathscr{T}_p|_U)| = |\Pi_0(\mathscr{E}_p|_U)| = 2$,
and in dimension $d = 0$ it is $\mathscr{T}_p = \mathscr{E}_p = \emptyset$.
Proof: For all $p \in \mathrm{i}ntM$ the strong LC structure provides a thickened cone $\mathscr{C}_p^{m(p)}$. Since $m(p) > 0$, $\mathscr{C}_p^{m(p)}$ contains always a neighborhood $U \ni p$ homeomorphic to a neighborhood $\phi_p^{-1}(U) \ni 0$ of the standard cone which in any dimension has the desired properties. □

At any interior point $p \in \mathrm{i}ntM$ the open exterior $\mathscr{E}_p$ and the open interior $\mathscr{T}_p$ of the cone $\mathscr{C}_p$ are locally topologically distinguishable for $d > 1$, indistinguishable for $d = 1$, and empty for $d = 0$. With a strong LC structure $\mathscr{T}_p|_U \neq \mathscr{E}_p|_U \forall U \ni p \iff d+1 > 2$, whence locally in any neighborhood $U \ni p$ the interior and exterior of $\mathscr{C}_p \cap U$ at p in U has an intrinsic invariant meaning. $\mathscr{C}_p|_U$ divides $U - \mathscr{C}_p|_U$ in three open submanifolds, a non-contractable exterior $\mathscr{E}_p|_U$, plus two contractable connected components of $\mathscr{T}_p =: \mathscr{F}_p|_U \cup \mathscr{P}_p|_U$, the local future $\mathscr{F}_p|_U$ and the local past $\mathscr{P}_p|_U$ with $\partial\mathscr{F}_p|_U = \mathscr{C}_p^+|_U$ where $\mathscr{C}_p^+ := (\phi_p \mathscr{C}^+)$ and $\partial\mathscr{P}_p|_U = \mathscr{C}_p^-|_U$ where $\mathscr{C}_p^- := \phi_p \mathscr{C}^-$ respectively. This rises also the question if and how $\mathscr{F}_p$ and $\mathscr{P}_p$ or their local restriction to $U \ni p$ can be distinguished. This problem is solved by a topological $\mathbb{Z}_2$ connection (see also Section 2.2 below).

Given a strong LC structure, a compatible local (conformal) metric can always be proven to exist on any differentiable manifold M with CAT$\subset \mathcal{C}^1$. Within such CAT, let η be a Lorentzian metric on $\mathbb{R}^{d+1}$ which is compatible with $\mathscr{C}$ by being at 0 (CAT-)asymptotically the flat Minkowskian one. It can be restricted to $\mathscr{C}^m$ and pulled back pointwise along $(\phi_p)^{-1}$ to a metric g on $\mathscr{C}_p^{m(p)}$. The CAT continuity of the family $\{p \mapsto \mathscr{C}_p^{m(p)}\}$ implies CAT$_{-1}$ continuity of the family $\{p \mapsto g|_{\mathscr{C}_p^{m(p)}}\}$. So we can extract a CAT$_{-1}$ continuous metric $\{p \mapsto g_p\}$.

Here we are interested particularly in Lorentzian metrics which are *locally compatible* with a (weak or strong) LC structure. The Minkowski metric η is locally compatible with the cone $\mathscr{C}$ in the sense that $\eta_0(v,v) = 0 \Leftrightarrow v \in T_0N$, with arbitrary submanifold $N \subset \mathscr{C} \subset \mathbb{R}^{d+1}$ such that $(0,v) \in TN$.

Definition: A Lorentzian metric g and a LC structure $p \mapsto \mathscr{C}_p$ on a given manifold M are said to be *locally compatible*, iff $\forall p \in \text{int}M$ and $\mathscr{C}_p \supset \phi_p(N) \cong N$ it holds:

$$g_p(V(p), V(p)) = 0 \Leftrightarrow V(p) \in T_p\phi_p(N) \ . \tag{2.5}$$

Obviously (2.5) just means that locally at any vertex the cone determines the characteristic null directions in the tangent space.

On the other hand, the cone structure poses an equivalence relation on Lorentzian metrics which are compatible with the LC structure. Given any such metric g, the corresponding equivalence class $[g]$ is the conformal class of g. We summarize the existence and uniqueness result as follows:

Proposition 2: Given a strong LC structure on a (CAT) manifold,
(i) there always exist a (CAT_{-1}) Lorentzian metric g on M compatible with the LC structure.
(ii) the conformal class $[g]$ of LC compatible metrics is uniquely determined by the LC structure.

The existence of a conformal Lorentzian metric is guaranteed by a *strong* LC structure, but not by a weak one. However, since conditions 2.7 needs only the existence of the tangent bundle of $\mathscr{C}_p$, uniqueness is assured already by a differentiable *weak* LC structure.

Although at each $p \in \text{int}M$ a CAT strong LC structure on M admits a conformal class $[g]$ of CAT_{-1} Lorentzian metrics g with characteristic directions in T_pM tangential to $\mathscr{C}_p$, away from the vertex p the cones of the LC structure need not at all be compatible with the null structure of any conformal metric $[g]$. This reflect the fact that, apart from its local vertex structure, a strong LC structure is still much more flexible than a conformal structure. For any $q \neq p$ the tangent directions given by $T_q\mathscr{C}_p$ need a priori not be related to tangent directions of null curves of g, since the cone (or its thickening) at p is in general unrelated to that at q. The need for compatibility conditions between cones at different points motivates the introduction of some of the causality structures in open regions of M introduced later in the following section.

2.2 Causality on topological manifolds

Given a (weak or strong) LC structure one wants to know first of all under which conditions, for given $p \in \text{int}M$ an exterior and interior of the cone can be distinguished within the complement $M - \mathscr{C}_p$. This problem is the global analogue of the local one which was answered by Proposition 1 and Theorem 1 above.

Proposition 3: Assume that at $p \in \text{int}M$ there are open (CAT) submanifolds $\mathscr{T}_p$ and $\mathscr{E}_p$ such that the interior of M decomposes into the disjoint union $\text{int}M = \mathscr{C}_p \mathring{\cup} \mathscr{T}_p \mathring{\cup} \mathscr{E}_p$. Assume $\exists\, k \in \mathbb{N}_0 : \Pi_k(\mathscr{T}_p) \neq \Pi_k(\mathscr{E}_p)$. Then $\mathscr{T}_p$ and $\mathscr{E}_p$ can be topologically distinguished.
Proof: $\text{int}M - \mathscr{C}_p = \mathscr{T}_p \mathring{\cup} \mathscr{E}_p$, and homotopy groups are topological invariants. □

In particular the conditions of Proposition 3 apply for $d+1 > 2$ in particular to all manifolds with the following topological structure:

Example 1: Let in any dimension $d := \dim M - 1 > 0$ at any $p \in \mathrm{i}ntM$ be $|\Pi_0(\mathscr{T}_p)| = 2$ and $\Pi_{d-1}(\mathscr{E}_p) = \Pi_{d-1}(S^{d-1})$, for $d > 1$ be $\Pi_{d-1}(\mathscr{T}_p) = 0$ and $|\Pi_0(\mathscr{E}_p)| = 1$, For $d = 1$ be $\Pi_{d-1}(\mathscr{T}_p) = \Pi_{d-1}(\mathscr{E}_p) = \Pi_0(S^0)$, i.e. $|\Pi_0(\mathscr{T}_p)| = |\Pi_0(\mathscr{E}_p)| = 2$, and in dimension $d = 0$ be $\mathscr{T}_p = \mathscr{E}_p = \emptyset$ at any $p \in \mathrm{i}ntM$. Then in particular $\mathscr{T}_p \ncong \mathscr{E}_p \iff d + 1 > 2$. The open exterior $\mathscr{E}_p$ and the open interior $\mathscr{T}_p$ of the cone $\mathscr{C}_p$ at any interior point $p \in \mathrm{i}ntM$ are topologically distinct for $d > 1$, indistinguishable for $d = 1$, and empty for $d = 0$.

In the case of Example 1, $\mathscr{C}_p$ divides $M - \mathscr{C}_p$ in three open submanifolds, a non-contractable exterior $\mathscr{E}_p$, plus two contractable connected components of $\mathscr{T}_p =: \mathscr{F}_p \cup \mathscr{P}_p$, the future $\mathscr{F}_p$ and the past $\mathscr{P}_p$ with $\partial \mathscr{F}_p = \mathscr{C}_p^+ := \phi_p \mathscr{C}^+$ and $\partial \mathscr{P}_p = \mathscr{C}_p^- := \phi_p \mathscr{C}^-$ respectively. This rises also the question if and how $\mathscr{F}_p$ and $\mathscr{P}_p$ can be distinguished.

Let M be differentiable and τ be any vector field $M \to TM$ such that at any $p \in \mathrm{i}ntM$ its orientation agrees with that of $\phi_p(a)$. Such a orientation vector field comes naturally along with a (CAT_{-1}) $\mathbb{Z}_2$-connection on M which allows to compare the orientation $\tau(p)$ at different $p \in \mathrm{i}ntM$. Given a strong LC structure on M, the $\mathbb{Z}_2$-connection is in fact provided via continuity of $p \mapsto T_p\phi_p(a)$. Then τ is tangent to an integral curve segment through p from $\mathscr{P}_p$ to $\mathscr{F}_p$. In particular, $\mathscr{F}_p$ and $\mathscr{P}_p$ are distinguished from each other by a consistent τ-orientation on M.

If M is not differentiable, in order to distinguish continuously $\mathscr{P}_p$ from $\mathscr{F}_p$ on $\mathrm{i}ntM$ it remains just to impose a topological $\mathbb{Z}_2$-connection on $\mathrm{i}ntM$ a fortiori.

In order to obtain a more specific causal structure it remains to identify natural consistency conditions for the intersections of cones of different points. In order to define topologically timelike, lightlike, and spacelike relations, and a reasonable causal complement, we introduce the following causal consistency conditions on cones.

Definition 8: M is *(locally) cone causal* or *C-causal* in an open region U, if it carries a (weak or strong) LC structure and in U the following relations between different cones in $\mathrm{i}ntM$ hold:

(1) For $p \neq q$ one and only one of the following is true:

(i) q and p are causally *timelike* related, $q \ll p :\Leftrightarrow q \in \mathscr{F}_p \wedge p \in \mathscr{P}_q$ (or $p \ll q$)

(ii) q and p are causally *lightlike* related, $q \lhd p :\Leftrightarrow q \in \mathscr{C}_p^+ - \{p\} \wedge p \in \mathscr{C}_q^- - \{q\}$ (or $p \,\mathscr{C}\, q$),

(iii) q and p are causally unrelated, i.e. relatively *spacelike* to each other, $q \bowtie p :\Leftrightarrow q \in \mathscr{E}_p \wedge p \in \mathscr{E}_q$.

(2) Other cases (in particular non symmetric ones) do not occur.

M is *locally C-causal*, if it is C-causal in any region $U \subset M$. M is *C-causal* if conditions (1) and (2) hold $\forall p \in \mathscr{C}$.

Let M be C-causal in U. Then, $q \ll p \Leftrightarrow \exists r : q \in \mathscr{P}_r \wedge p \in \mathscr{F}_r$, and $q \lhd p \Leftrightarrow \exists r : q \in \mathscr{C}_r^+ \wedge p \in \mathscr{C}_r^-$.

If an open curve $\mathbb{R} \ni s \mapsto c(s)$ or a closed curve $S^1 \ni s \mapsto c(s)$ is embedded in M, then in particular its image is $\mathrm{i}m(c) \equiv c(\mathbb{R}) \cong \mathbb{R}$ or $\mathrm{i}m(c) \equiv c(S^1) \cong S^1$ respectively, whence it is free of self-intersections. Such a curve is called *spacelike* $:\Leftrightarrow \forall p \equiv c(s) \in \mathrm{i}m(c) \exists \epsilon : c|_{]s-\epsilon, s+\epsilon[-\{s\}} \in \mathscr{E}_{c(s)}$, and *timelike* $:\Leftrightarrow \forall p \equiv c(s) \in \mathrm{i}m(c) \exists \epsilon : c|_{]s-\epsilon, s+\epsilon[-\{s\}} \in \mathscr{T}_{c(s)}$.

Note that C-causality of M forbids a multiple refolding intersection topology for any two cones. In particular along any timelike curve the future/past cones do not intersect, because otherwise there would exist points which are simultaneously timelike and lightlike related. Continuity then implies that future/past cones in fact foliate the part of M which they cover. Hence, if there exists a fibration $\mathbb{R} \hookrightarrow \mathrm{i}ntM \;\; \Sigma$, then C-causality implies that the future/past cones foliate in particular on any fiber. In fact, given a fibration, one could define also a weaker form of causality just by the foliating property of all future/past cones on each fiber. (Physically this situation corresponds to ultralocal classical clocks. Quantum uncertainty of the fiber would require to take appropriate ensemble averages over some bundle of neighboring fibers which then contains in particular spacelike related vertices on the fibers of the bundle. Then the corresponding future or past cones intersect for sure, and even timelike related ones of different fibers *may* intersect !) C-causality however requires more, namely the future/past cones of *all* timelike related vertices should be non-intersecting, not only those in a particular fiber.

Therefore C-causality allows also a reasonable definition of a causal complement.

Definition 9: For any open set S in a C-causal manifold M the *causal complement* is defined as

$$S^{\perp} := \bigcap_{p \in \mathrm{cl}S} \mathscr{E}_p, \tag{2.6}$$

where cl$S$ denotes the closure in the topology induced from the manifold. Although the causal complement is always open, it will in general not be a contractable region even if S itself is so.

Assume p and q are timelike related, $p \in \mathscr{P}_q$ and $q \in \mathscr{F}_p$. $\mathscr{K}_p^q := \mathscr{F}_p \cap \mathscr{P}_q$ is the bounded open double cone between p and q. Given any bounded open $\mathscr{K}$ such that $\exists p, q \in M : \mathscr{K} = \mathscr{F}_p \cap \mathscr{P}_q$, we set $i^+(\mathscr{K}) := \{q\}$, $i^-(\mathscr{K}) := \{p\}$, and $i^0(\mathscr{K}) := \mathscr{C}_p^+ \cap \mathscr{C}_q^-$. For any $\mathscr{K}_p^q \subset M$ let $\mathrm{cl}_c(\mathscr{K}_p^q)$ be its *causal closure*.

Since C-causality prohibits relative refolding of cones, it also ensures that $(\mathscr{K}_p^q)^{\perp\perp} = \mathscr{K}_p^q$, i.e. the causal complement is a duality operation on double cones.

The open double cones of a C-causal manifold M generate a topology, called the *double cone topology* which is a genuine generalization of the usual Alexandrov topology for strongly causal space-times. For strongly causal space-times the Alexandrov topology is equivalent to the manifold topology [46, 47]. When M fails to be locally causal the double cone topology may be coarser than the manifold topology.

Let us discuss now possible natural relations that can appear between two double cones $\mathscr{K}_1$ and $\mathscr{K}_2$ of a C-causal manifold. First there is the case $\mathscr{K}_1 \cap \mathscr{K}_2 = \emptyset$ which corresponds to causally unrelated sets. For $\mathscr{K}_1 \cap \mathscr{K}_2 \neq \emptyset$, the intersection is such that $\mathscr{K}_1 \cup \mathscr{K}_2 - \mathscr{K}_1 \cap \mathscr{K}_2$ is either given by two disconnected pieces or it is connected. In the latter case we distinguish whether $\partial\mathscr{K}_1 \cap \partial\mathscr{K}_2$ is empty or not. It is in the former case that one of $\mathscr{K}_1$ and $\mathscr{K}_2$ will be contained in the other.

Local C-causality does a priori not preclude other more pathological possibilities. However it is possible to define in a purely topological manner more refined causality notions.

Definition 10: Let M be a C-causal manifold.

(i) M is *globally hyperbolic* $:\Leftrightarrow \mathrm{cl}_c\mathscr{K}_p^q$ compact $\forall p, q \in M$

(ii) M is *causally simple* $:\Leftrightarrow \mathrm{cl}_c\mathscr{K}_p^q$ closed $\forall p, q \in M$

(iii) M is *causally continuous* $:\Leftrightarrow M$ is distinguishing and both $\mathscr{F} : p \mapsto \mathscr{F}_p$ and $\mathscr{P} : q \mapsto \mathscr{P}_q$ are continuous

(iv) M is *stably causal* $:\Leftrightarrow$ M admits a $\mathcal{C}^0$ function $f: M \to \mathbb{R}$ strictly monotoneously increasing along each future directed nonspacelike curve (global time function)
(v) M is *strongly causal* $:\Leftrightarrow$ the topology generated by $\{\mathscr{K}_p{}^q\}_{p,q\in M}$ is equivalent to the manifold topology of M
(vi) M is *distinguishing* $:\Leftrightarrow \mathscr{F}_p = \mathscr{F}_q \Rightarrow p = q \wedge \mathscr{P}_p = \mathscr{P}_q \Rightarrow p = q$
(vii) M is *causal* $:\Leftrightarrow$ every closed curve in M is not nonspacelike
(viii) M is *chronological* $:\Leftrightarrow$ every closed curve in M is not timelike

If a manifold carries a Lorentzian metric, we saw in Section 2.1 above that this is locally compatible with a strong LC structure. Beyond that, it is an interesting question under which conditions a Lorentzian metric is *compatible* with some LC structure. The Minkowski metric η is compatible with the cone $\mathscr{C}$ in the sense that $\eta_x(v,v) = 0 \Leftrightarrow (x,v) \in T\mathscr{C} := \bigcup_{y\in\mathscr{C}} T_y\mathscr{C}$ where $T_y\mathscr{C} := \bigcup_{y\in N\subset\mathscr{C}} T_yN \subset \mathbb{R}^{d+1}$ and the latter union is over all (differentiable) 1-dimensional submanifolds $N \subset \mathscr{C}$ passing through y, with all their tangent spaces embedded as linear submanifolds with common origin within the common embedding space $\mathbb{R}^{d+1}$. Hence, for $y \neq 0$, the fibers $T_y\mathscr{C} \cong \mathbb{R}^d$ are all usual isomorphic tangent spaces, while the only non-standard fiber $T_0\mathscr{C} \cong \mathscr{C} \subset \mathbb{R}^{d+1}$ reproduces the d-dimensional cone itself, which is the local model of its own singularity.

Definition: A Lorentzian metric g and a LC structure $p \mapsto \mathscr{C}_p$ on some manifold M are said to be *compatible*, iff $\forall q \in \mathrm{int}M$ it holds:

$$g_q(V(q),V(q)) = 0 \;\Leftrightarrow$$
$$\left[\forall p \in M : q \in \mathscr{C}_p \Rightarrow V(q) \in T_q\mathscr{C}_p := (\phi_p)_* T_{\phi_p{}^{-1}(q)}\mathscr{C} = \bigcup_{\phi_p{}^{-1}(q)\in N\subset\mathscr{C}} T_q\phi_p(N)\right] \quad ,(2.7)$$

where the latter union is over all (differentiable) 1-dimensional submanifolds $N \subset \mathscr{C}$ passing through $\phi_p{}^{-1}(q)$, and the latter identity holds with tangent push forward $(\phi_p)_*T_yN := T_{\phi_p(y)}\phi_p(N)$.

With (2.7) the cones are the (conformally) characteristic null surfaces of the Lorentzian metric. As pointed out above, (2.7) does not hold in general. However one might search for sufficient and necessary causality conditions such that this compatibility holds. A systematic investigation of this point is an interesting topic for further investigations. Let us here just assure the correspondence of the causality notions of Def. 10 to the usual ones in the case of a Lorentzian space-time.

Theorem 2: Let M carry a smooth Lorentzian metric g. Then the Lorentzian metric determines a C-causal structure. If a C-causal structure of M is related to a Lorentz metric, then the definitions (i)-(viii) agree with the standard definitions and the following chain of implications of properties of M holds: globally hyperbolic $\Rightarrow$ causally simple $\Rightarrow$ causally continuous $\Rightarrow$ stably causal $\Rightarrow$ strongly causal $\Rightarrow$ distinguishing $\Rightarrow$ causal $\Rightarrow$ chronological.

Proof: Given a smooth Lorentzian metric g the cones determined by the null structure $[g]$ respect the relations of Def. 8, because otherwise there would exist some singularities. For (v) in the case of Lorentzian manifolds see [47], for the other notions and for the chain of implications see [48]. □

Finally let us define a condition which excludes the existence of singularities or internal boundaries within the future and past cones.

Definition 11: Let M carry a (weak or strong) LC structure.
(i) M is *precausal* in an open region $U \subset M$, if the $d+1$-parameter CAT family $\{\phi_p\}_{p\in U}$ of CAT homeomorphisms $\phi_p : \mathbb{R}^{d+1} \supset V \to U$ is such that at any $p \in U$ it is $\mathscr{C}_p|_U = \phi_p \mathscr{C}|_V$, and any CAT submanifold of $\mathscr{C}_p$ or $(M - \mathscr{C}_p) \cap U$ is a CAT homeomorphic image of $\mathscr{C}$ or $(\mathbb{R}^{d+1} - \mathscr{C}) \cap V$ respectively. M is *locally precausal* iff it is precausal in any open region $U \subset M$.
(ii) M is *precausal* if it is locally precausal such that in the CAT $d+1$-parameter family $\{\phi_p\}_{p\in U}$ any CAT homeomorphism extends also to a homeomorphism of the interior $\phi_p : \mathscr{E} \to \mathscr{E}_p$.

2.3 Causal index sets and diffeomorphisms

Let us now define the index sets which will be used in our nets of algebras. The natural numbers $\mathbb{N}$ are the most common index set for any countable set on which they induce then a canonical order relation. However, in the following we consider more general index sets which need not be countable.

Definition: A *net index set* is an index set I (i) with a partial order $\leq$, (ii) with a sequence of $K_i \in I$, $i \in \mathbb{N}$, such that $\forall K \in I \exists j \in \mathbb{N}$: $K \leq K_j$, and (iii) such that each bounded $J \subset I$ has a unique supremum $\sup J \in I$.
Remark 1: If I is totally ordered (iii) is satisfied trivially.
Remark 2: By (ii) a net index set is infinite unless $\exists j \in \mathbb{N}$: $K_i = K_j \ \forall i \geq j$.

Definition: A *causal disjointness relation* in a net index set I is a symmetric relation $\perp$ such that

(i) $K_1 \perp K_0 \wedge K_2 < K_1 \Rightarrow K_2 \perp K_0$,
(ii) for any bounded $J \subset I$: $K_0 \perp K \ \forall K \in J \Rightarrow K_0 \perp \sup J$,
(iii) $\forall K_1 \in I \ \exists K_2 \in I$: $K_1 \perp K_2$.
A *causal index set* $(I, \perp)$ is a net index set with a causal disjointness relation $\perp$.

Definition: Let M be infinite with causal complement $\perp$. M is $\perp$*-nontrivially inductively covered*, iff $\exists$ sequence of nonempty $K_i \subset M$, $i \in \mathbb{N}$, mutually different with $(K_i)^\perp \neq \emptyset$ such that $\bigcup_{i=1}^\infty K_i = M$.

Example 2: Any conformal class of a Lorentzian metric, which is globally hyperbolic without any singularities determines such a causal structure. In this case the compact open double cones form a basis of the usual Euclidean $d+1$ topology. Each open compact double cone $\mathscr{K}$ is conformally equivalent to a copy of Minkowski space. Consider a spatial Cauchy section Σ of M and a geodesic world line $p : \tau \to M$ intersecting Σ at $p(0)$, where τ is the proper time of the observer. Now for any $\tau > 0$ the causal past of $p(\tau)$ intersects Σ in an open set O_τ. Then these open sets are totally ordered by their nested inclusion in Σ, and their order agrees also with the total order of worldline proper time,

$$O_{\tau_1} \subset O_{\tau_2} \Leftrightarrow \tau_1 < \tau_2. \tag{2.8}$$

This is the motivation to consider the partial order related to the flow of time and the one related to enlargement in space to be essentially the same, such that in the absence of an a priori notion of a metric time, the nested spatial inclusion will provide a partial order substituting time. (Of course this is in essence similar to the old idea in cosmology of time given by the volume of a closed, expanding universe.)

Consider now a double cone $\mathscr{K}$ in M with $O := \mathscr{K} \cap \Sigma$ and $\partial O = i^0(\mathscr{K})$ and a diffeomorphism ϕ in M with pullbacks $\phi^\Sigma \in \mathrm{Diff}(\Sigma)$ to Σ and $\phi^{\mathscr{K}} \in \mathrm{Diff}(\mathscr{K})$ to $\mathscr{K}$. If $\phi(\mathscr{K}) = \mathscr{K}$, it can be naturally identified with an element of $\mathrm{Diff}(\mathscr{K})$. ($\phi = id_{M-\mathscr{K}}$ is a sufficient but not necessary condition for that to be true.) If in addition $\phi(\Sigma) = \Sigma$ then also $\phi(O) = O$, and $\phi|_O$ is a diffeomorphism of O.

Let us now consider a 1-parameter set of double cones $\{\mathscr{K}_p\}$ sharing 2 common null curve segments $n_\pm \in \partial\mathscr{K}_p$ from $i^\pm(\mathscr{K}_p)$ respectively to $i \in i^0(\mathscr{K}_p)$ which they intersect transversally in Σ. Let such cones be parametrized by a line c in Σ starting (transversally to n) at i to some endpoint f on $\partial\Sigma$ (at spatial infinity) such that p is an interior point of $O_p := \mathscr{K}_p \cap \Sigma$. Then we call the limit $W(n_\pm, c) := \lim_{p\to f} K_p$ the wedge in the surface through $n_\pm$ and c. Note that in the usual (say Minkowski) metric case a wedge has a quite rigid structure, because c has a canonical location in a surface spanned by $n_\pm$. The present diffeomorphism invariant analogue is of course much less unique in structure.

3. Cosmological application I: quantum general relativity

This section deals with the application of the general, geometry-independent definitions of causal structure of the previous section.

Sect. 3.1. introduces an axiomatic framework of algebraic quantum field theory (AQFT) adapted to general relativistic quantum (field) theories. Sect. 3.2. then discusses quantum geometry within this setting.

3.1 Algebraic QFT on manifolds

Clearly QFT on a globally hyperbolic space-time manifold satisfies isotony (N1), covariance (N2), causality (C), additivity (A) and existence of a (state dependent GNS) vacuum vector (V). More particular on Minkowski space there is is a unique Poincare-invariant state ω such that there is a translational subgroup of isometries with spectrum in the closure of the forward light cone only. However there is no reason to expect such features in a more general context. However, an invariant GNS vacuum vector Ω still exists for a globally hyperbolic space-time, although in general it depends on the choice of the state ω. Hence we will now generalize the axioms of AQFT from globally hyperbolic space-times to differentiable manifolds.

For a given QFT on manifolds, say the example of quantum gravity examined below, it remains to check which of the generalized axioms will hold true.

3.1.1 General axioms for QFT on a differentiable manifold

On a differentiable manifold M part of the AQFT structure can be related to the topological structure of M. The following AQFT axioms are purely topological and should hold on arbitrary differentiable manifolds. Let M be a differentiable manifold with additional structure s (which may be empty) and $\mathrm{Diff}(M,s)$ denote all diffeomorphisms which preserve s. A $\mathrm{Diff}(M,s)$-invariant algebraic QFT (in the state ω) can be formulated in terms of axioms on a net of $*$-algebras $\mathcal{A}(\mathcal{O})$ (together with a state ω thereon). It should at least satisfy the following axioms:

N1 (Isotony):

$$\mathcal{O}_1 \subset \mathcal{O}_2 \quad \Rightarrow \quad \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \; \forall \mathcal{O}_{1,2} \in \mathrm{Diff}(M,s) \tag{3.1}$$

N2 (Covariance):

$$\mathrm{Diff}(M,s) \ni g \overset{\exists}{\mapsto} U(g) \in U(\mathrm{Diff}(M,s)) \quad : \quad \mathcal{A}(g\mathcal{O}) = U(g)\mathcal{A}(\mathcal{O})U(g)^{-1} \; . \tag{3.2}$$

(N1) and (N2) are purely topological, involving only the mere definition of the net. These axioms make sense even without a causal structure (see also [5]).

If $\mathcal{A}(\mathcal{O})$ is a C^*-algebra with norm $||\cdot||$, it makes sense to impose the following additional axioms:

A (additivity):

$$\mathcal{O} = \cup_j \mathcal{O}_j \quad \Rightarrow \quad \mathcal{A}(\mathcal{O}) = \mathrm{cl}_{||\cdot||}\left(\cup_j \mathcal{A}(\mathcal{O}_j)\right) \; . \tag{3.3}$$

V (Invariant Vacuum Vector): Given a state ω, there exists a representation π_ω on a Hilbert space $\mathcal{H}_\omega$ such that

$$\begin{aligned} &\exists \Omega \in \mathcal{H}_\omega, ||\Omega|| = 1 \quad : \\ &\qquad \text{(cyclic)} \qquad \left(\cup_{\mathcal{O}} \mathcal{R}(\mathcal{O})\right) \Omega \overset{\text{dense}}{\subset} \mathcal{H}_\omega \\ &\qquad \text{(invariant)} \qquad U(g)\Omega = \Omega \; , \quad g \in \mathrm{Diff}(M,s) \; . \end{aligned} \tag{3.4}$$

Note: For any $*$-algebra, the representation π_ω is given by the GNS construction, $\mathcal{H}_\omega$ is the GNS Hilbert space. Properties of Ω are induced by corresponding properties of the state ω. The main issue to check is the invariance under a unitary representation U of Diff (M,s).

3.1.2 Axioms for QFT on a manifold with cone causality

With a notion of causality on a differentiable manifold M as defined in the previous section, the algebraic structure of a QFT should be related to the causal differential structure of M by further axioms abstracted from the space-time case. In this case it is natural to consider nets of von Neumann algebras. On a causal differential manifold M (in the sense defined above) the algebraic structure of a QFT should satisfy the following axioms which require the notion of a causal complement. Let M be a causal differentiable manifold with additional structure s (which may be empty) and $\mathrm{Diff}(M,s)$ denote all differentiable diffeomorphisms which preserve s, where s is at least a causal structure, eventually with some additional structure s'. A Diff(M, s)-invariant algebraic QFT in the state ω is a net of von Neumann-algebras $\mathcal{R}(\mathcal{O})$ with a state ω satisfying the following axioms:

C (causality):

$$\mathcal{O}_1 \perp \mathcal{O}_2 \quad \Rightarrow \quad \mathcal{R}(\mathcal{O}_1) \subset \mathcal{R}(\mathcal{O}_2)' \ . \tag{3.5}$$

CA (causal additivity):

$$\mathcal{O} = \cup_j \mathcal{O}_j \quad \Rightarrow \quad \mathcal{R}(\mathcal{O}) = (\cup_j \mathcal{R}(\mathcal{O}_j))'' \ . \tag{3.6}$$

Remarks: In the case that the net has both inner and exterior boundary, (3.5) had been weekened in [5] to a generalization of Haag duality on the boundary of the net. Here we do not assume a priori the existence of such a boundary of the net. However an example of quantum geometry with such a boundary structure is discussed below.

Given a net of C^* algebras consistent with a norm $||\cdot||$, it made sense to impose (A) above. If the algebras are in particular also von Neumann ones (A) should be sharpened to (CA). In the general case of $*$-algebras (not necessarily C^* ones) the algebraic closure has no natural topological analogue, and hence there is no obvious definition of additivity. Therefore in [5] neither (A) nor (CA) was assumed.

3.2 Quantum geometry

On a region causally exterior to a topological horizon $\mathscr{H}$, on any d-dimensional spatial slice Σ, there exists a net of Weyl algebras for states with an *infinite* number of intersection points of edges and transversal $(d-1)$-faces within any neighbourhood of the spatial boundary $\mathscr{H} \cap \Sigma \cong S^2$.

Σ be a spatial slice. C-causality constrains the algebras localized within Σ. On Σ it should hold

$$\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset \quad \Rightarrow \quad [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0. \tag{3.7}$$

A (spin network) state ω over the algebra $\mathcal{A}(\Sigma)$ may be defined by a closed, oriented differentiable graph γ embedded in Σ, with an infinite number of differentiable edges $e \in E$ intersected transversally by a differentiable $d-1$-dimensional oriented surface S at a countable of intersection vertices $v \in V$. Let $C_\gamma \in \mathrm{Cyl}^r$ be a $\mathcal{C}^\infty$ Cylinder function with respect to an SO(d) holonomy group on γ, i.e. on each closed *finite* subgraph $\gamma' \subset \gamma$ it is $C_{\gamma'} := c(g_1, \ldots, g_N)$ where $g_k \in \mathrm{SO}(d)$, and c is a differentiable function. With test function f the action of a derivation $X_{S,f}$ on Cyl^r is defined by

$$X_{S,f} \cdot C_\gamma := \frac{1}{2} \sum_{v \in V} \sum_{e_v \in E : \partial e_v \ni v} \kappa(e_v) f^i(v) X^i_{e_v} \cdot c, \tag{3.8}$$

where $\kappa(e_v) = \pm 1$ above/below S (for the following purposes we may just exclude the tangential case $\kappa(e_v) = 0$) and $X^i_{e_v} \cdot c$ is the action of the left/right invariant vector field (i.e. e_v is oriented away from/towards the surface S) on the argument of c which corresponds to the edge e_v. Let $\mathrm{Der}(\mathrm{Cyl}^r)$ denote the span of all such derivations.

Here the classical (extended) phase space is the cotangent bundle $\Gamma = T^*\mathcal{C}$ over a space $\mathcal{A}$ of (suitably regular) finitely localized connections. Let $\delta = (\delta_A, \delta_E) \in T_e\Gamma$. With suitable boundary conditions, a (weakly non-degenerate) symplectic form Ω over Γ acts via

$$\Omega|_{(A_e,E_e)}(\delta,\delta') \;:=\; \frac{1}{\ell^2}\int_\Sigma \mathrm{Tr}\,[*E\wedge A' - *E'\wedge A]\,. \tag{3.9}$$

After lifting from $\mathcal{C}$ to Γ, the cylinder functions $q \in \mathrm{Cyl}^r$ serve as (gauge invariant) classical elementary configuration functions on Γ. The derivations $p \in \mathrm{Der}(\mathrm{Cyl}^r)$ serve as classical elementary momentum functions on Γ. They are obtained as the Poisson-Lie action of 2-dimensionally smeared duals of densitized triads E. $\mathrm{Cyl}^r \times \mathrm{Der}(\mathrm{Cyl}^r)$ has a Poisson-Lie structure

$$\{(q,p),(q',p'))\} := (pq' - p'q, [p,p']), \tag{3.10}$$

where $[p,p']$ denotes the Lie bracket of p and p'. An antisymmetric bilinear form on $\mathrm{Cyl}^r \times \mathrm{Der}(\mathrm{Cyl}^r)$ is given by

$$\Omega_0\left((\delta_q,\delta_p),(\delta'_q,\delta'_p)\right) \;:=\; \int_{\mathcal{C}_{\gamma\cup\gamma'}/\mathcal{G}_{\gamma\cup\gamma'}} d\mu_{\gamma\cup\gamma'}\,[pq' - p'q]\,, \tag{3.11}$$

where $q, q' \in \mathrm{Cyl}^r$ have support on γ resp. γ', with $pq' - p'q \in \mathrm{Cyl}^r$ integrable over $\mathcal{C}_{\gamma\cup\gamma'}/\mathcal{G}_{\gamma\cup\gamma'}$ with measure $d\mu_{\gamma\cup\gamma'}$.

On $T_e\Gamma$, the symplectic form Ω yields functions of the form $\Omega((\delta_A,\delta_E),\cdot)$. Canonical quantization then associates to any function $\Omega(f,\cdot)$ a selfadjoint operator $\hat{\Omega}(f,\cdot)$ and a corresponding unitary Weyl element $W(f) := e^{i\hat{\Omega}(f,\cdot)}$, both on some extended Hilbert space. With multiplication $W(f_1)W(f_2) := e^{i\Omega(f_1,f_2)}W(f_1+f_2)$, and conjugation $* : W(f) \mapsto W(-f)$, the Weyl elements generate a $*$-algebra. A norm on Γ is defined by $\|f\| := \frac{1}{4}\sup_{g\neq 0}\frac{\Omega(f,g)}{<g,g>}$. The C^*-closure under the sup-norm then generates a C^*-algebra $CCR(W(f),\Omega)$. With regular Ω this CCR Weyl algebra is simple, i.e. there is no ideal. Observables of quantum 3-geometry are then the selfadjoint elements within a gauge and 3-diffeomorphism invariant C^*-subalgebra $\mathcal{A}_\gamma \subset C^*(W(f), f \in \Gamma)$. In a gauge and 3-diffeomorphism invariant representation of $\mathcal{A}_\gamma$, typical observables in are represented by configuration multiplication operators $C_\gamma \in \mathrm{Cyl}^r$ on Hilbert space $\mathcal{H}_\gamma$, and by gauge-invariant and 3-diffeomorphism invariant combinations of derivative operators $X_{S,f} \in \mathrm{Der}(\mathrm{Cyl}^r)$, like e.g. a certain quadratic combination which yields the area operator.

For each finite $\gamma' \subset \gamma$, the sets $E(\gamma')$ and $V(\gamma')$ of edges resp. vertices of γ' are finite. Then the connections $\mathcal{C}_{\gamma'} = \prod_{e\in E(\gamma')} G_e \cong G^{E(\gamma')}$ and the gauge group $\mathcal{G}_{\gamma'} = \prod_{v\in V(\gamma')} G_v \cong G^{V(\gamma')}$ on γ' inherit a unique measure from the measure on G (for compact G the Haar measure). The action of $\mathcal{G}_{\gamma'}$ on $\mathcal{C}_{\gamma'}$ is defined by $(gA)_e := g_{t(e)}A_e g^{-1}_{s(e)}$ where s and t are the source and target functions $E(\gamma') \to V(\gamma')$ respectively. This action gives rise to gauge orbits and a corresponding projection $\mathcal{C}_{\gamma'}\;\; \mathcal{C}_{\gamma'}/\mathcal{G}_{\gamma'}$. The projection induces the measure on $C_{\gamma'}/\mathcal{G}_{\gamma'}$. Bounded functions w.r.t. to that measure define then the gauge invariant Hilbert space $\mathcal{H}_{\gamma'} := L(\mathcal{C}_{\gamma'}/\mathcal{G}_{\gamma'})$.

However, over finite graphs, all is still QM rather than QFT. In order to obtain an infinite number of degrees of freedom on any finite localization domain which includes the inner boundary S^{d-1} (the intersection S^{d-1} of $\mathscr{H}$ and Σ), let S^{d-1} be intersected by an infinite number of edges of some graph γ in the exterior spatial neighborhood of $\mathscr{H}$. In the 3+1-dimensional case, evaluation of the area operator on the puncture of the boundary S^2 from edge p yields a quantum of area proportional to $j_p(j_p+1)$ for edge p carrying a spin-j_p representation of the group G. Since S^2 is compact the punctures should have at least one accumulation point. Hence for typical configurations in the principal representation, near that accumulation point the area will explode to infinity. When almost all punctures are located in arbitrary small neighborhoods of a finite number n of accumulation points, corresponding states represent quantum geometries of a black hole with n stringy hairs extending out to infinity. In particular, the $n = 1$ case was discussed in more detail in [9].

4. Homogeneous structures

4.1 Homogeneous manifolds

A Hausdorff (i.e. T_2-strongly separating) topological or differentiable manifold M is topologically homogeneous, per definition, iff all local neighborhoods of all points are homeomorphic. This means in particular that each point $p \in M$ has to be an interior point $p \in \mathrm{i}ntM$. In other words, the manifold is already given by the set of its interior points, $M = \mathrm{i}ntM$, i.e. the manifold is either entirely open or closed without boundary, $M \cap \partial M = \emptyset$. In a homogeneous topological manifold any point contains an open region, i.e. a neighborhood homeomorphic to an open ball of dimension $\dim M$.

Note that a homogeneous C^r-differentiable structure on a closed manifold need not be C^{r+1}-homogeneous, while the vice versa is always true.

A geometry (of arbitrary signature) is homogeneous iff the intrinsic curvature at all points is the same, which is the case if the connection is a homogeneous one.

The homogeneity of a manifold M can equivalently be expressed by the existence of a transitive group action on the manifold. A manifold is characterized by its homeomorphism group. If M is homogeneous, any two points are connected by a local homeomorphism, and the (local) homeomorphism group $\mathrm{Hom}M$ acts on the entire manifold. (Note: Here we do not consider global homeomorphisms since this would be a topic in its own.) If M contains boundary points, it decomposes non trivially into interior $\mathrm{i}ntM$ and collared boundary $\mathbb{R}^+ \times \partial M$, which intersect in an open tube $\mathbb{R} \times \partial M$. The local topological model of a collared boundary point reduces here essentially to the half line $\mathbb{R}^+$ (i.e. a 1-valent vertex). The homeomorphism group $\mathrm{Hom}M$ then decomposes into homomorphisms group $\mathrm{Hom}(\mathrm{i}ntM)$ of the interior and homeomorphisms of the collared boundary $\mathrm{Hom}(\mathbb{R}^+ \times \partial M)$. $\mathrm{Hom}(\mathrm{i}ntM)$ acts transitively on $\mathrm{i}ntM$, whence $\mathrm{i}ntM$ is homogeneous. Besides, $\mathrm{Hom}(\mathrm{i}ntM)$ and $\mathrm{Hom}(\mathbb{R}^+ \times \partial M)$ both act also non-transitively on the collared boundary which determines the inhomogeneous part of the global topology.

Similarly for a more general stratifiable variety V, the homeomorphism group contains the subgroup $\mathrm{Hom}(\mathrm{i}ntV)$ acting transitively on its interior, which is the piece made up by all points with open neighborhoods homeomorphic to the open cell of generic dimension. $\partial V := V - \mathrm{i}ntV$ can then be collared within V. The resulting collaring will then consists from vertices of different valences. In particular 0-valent vertices then belong to points of ∂V which are not limit points of $intV$ and 1-valent vertices to traditional boundary points, 2-valent vertices will play no role in the topological category, but appear in differentiable categories within kinks of V, and similarly, vertices of valence $n \geq 3$ appear at the join of n leaves within V. Again the collaring of ∂V contains the inhomogeneous part of the global topology.

Topological spaces with homogeneous (Hausdorff) dimension are invariant under rather general homeomorphisms. Restricting to the interior always yields topological homogeneity.

However, if the manifold carries more structure, homeomorphisms have to preserve not only the local topology but also the additional structure. So, a C^r-differential structure on a manifold restricts the homeomorphism to C^r-diffeomorphisms. Similarly, a causal structure on a manifold restricts its homeomorphisms to those which preserve the causal structure.

Definition: A Riemannian manifold (M, g) (of arbitrary signature) is a manifold M equipped with a symmetric bilinear $\mathcal{C}^\infty$ section $g : M \to \mathfrak{T}^0_2 M$ called metric. Unless specified otherwise the metric g will always be assumed to be non-degenerate.

A diffeomorphism of a Riemannian manifold (M, g) is a structure homeomorphism of the latter, iff it is an isometry.

The very fact that a given diffeomorphism $\chi \in \mathrm{Diff}(M)$ may be an isometry on some metric but not on another one is the reason why the action of $\mathrm{Diff}(M)$ is not free on the space $\mathrm{Met}(M)$ of $\mathcal{C}^\infty$-metrics on M, whence $\mathrm{Geom}(M) := \mathrm{Met}(M)/\mathrm{Diff}(M)$ is in general not a manifold.

Structure preserving homeomorphisms form a group. A structure s on a manifold M is homogeneous, iff the structure preserving homeomorphisms group $\mathrm{Hom}(M, s)$ acts transitively on M. For a (pseudo-)Riemannian metric, $s = g$, the homeomorphism group $\mathrm{Hom}(M, g)$ is the isometry group.

Definition: A Riemannian manifold (M, g) is called homogeneous, whenever M is homogeneous with a corresponding group G having a transitive realization $\tau(G) \subset \mathrm{Diff}(M)$ which leaves g invariant, i.e.

$$g_{\chi(p)} = g_p \quad \forall p \in M \quad \forall \chi \in \tau(G). \tag{4.1}$$

The classification of local homogeneous manifolds of given dimension is a clue to a systematic understanding of their possible deformation into each other.

Since the local isometry subgroups of a local homogeneous Riemannian manifold are Lie groups, it is useful, before trying to classify the homogeneous manifolds, to find first all possible contractions and, more generally, all possible limit transitions between real (or complex) Lie algebras of fixed dimension, and to uncover the natural topological structure of the space of all such Lie algebras.

Once the structure of the classifying space K^n is known, this information can be used as a first ingredient to construct the space of local Riemannian n-manifolds. This is demonstrated explicitly for $n = 3$ below.

4.2 Local homogeneous geometries

This section reviews local features of homogeneous geometries of general signature Riemannian type. Here we want to investigate the data which characterizes the *local* structure of a homogeneous Riemannian or pseudo Riemannian manifold (M, g). If we consider for arbitrary dimension n the different possible signatures modulo the reflection $g \to -g$, then $1+[\frac{n}{2}]$ different signature classes are distinguished by the codimension $s = 0, \ldots, n-[\frac{n}{2}]$ of the characteristic null hypersurface in the tangent space. For Lorentzian signature $s = 1$ the latter is an $(n-1)$-dimensional open double cone at the base point, in the Riemannian case $s = 0$ it is just the base point itself.

Per definition, a homogeneous manifold admits a transitive action of its isometry group. Let us restrict here to the case where it has even more a simply transitive subgroup of the isometry group. In this case we can solder the metric to an orthogonal frame spanned by the Lie algebra generators e_i in the tangent space, i.e.

$$g_{\mu\nu} = e^a_\mu e^b_\nu g_{ab} \tag{4.2}$$

where $e^a = e^a_\mu dx^\mu = g^{ai} e_i$, $e_i = e^\mu_i \frac{\partial}{\partial x^\mu}$, $g^{ab} g_{ij} = \delta^a_i \delta^b_j$, with the constant metric

$$(g_{ab}) = \begin{bmatrix} \epsilon_1 e^s & 0 & 0 \\ 0 & \epsilon_2 e^{s+w-t} & 0 \\ 0 & 0 & \epsilon_3 e^{s-t} \end{bmatrix}. \tag{4.3}$$

Here s fixes the overall scale, while t and w parametrize the anisotropies related respectively to the e_1 and e_2 direction (maintaining isotropy in the respective orthogonal planes).

The local data can be rendered in form of (i) the *local scales* of (4.3), (ii) the *covariant derivatives*

$$De^k = e^k_{i;j} e^i e^j := e^k_{\alpha;\beta} dx^\alpha dx^\beta \tag{4.4}$$

of the dual generators e^k in the cotangent frame, (iii) the corresponding *Lie algebra*

$$[e_i, e_j] = C^k_{ij} e_k, \tag{4.5}$$

and (iv) the *orientation* of the $n-s$-dimensional null hyperspace in the tangent space. For the Lorentzian case $s = 1$, this orientation is described by the future oriented normal vector n along the central axis of the double cone,

$$n = n^a e_a, \tag{4.6}$$

where its triad frame components n^a have to be coordinate independent, since the manifold is assumed to be homogeneous.

Let us consider now $n = 3$. In this case, there are no further signature cases besides the Riemannian and Lorentzian ones.

In the Riemannian case, the datum (iv) is trivial. For this case we give the complete classification of local homogeneous 3-spaces with some isometry subgroup in K^3 below.

Here, the Kantowski-Sachs (KS) spaces, here the only exception not admitting a simply-transitive subgroup of their isometry group, can be obtained as a specific limits of Bianchi IX spaces. The global geometrical correspondence of such a limit is given by a hyper-cigar like 3-ellipsoid of topology S^3, stretched infinitely long to become a hyper-cylinder $S^2 \times \mathbb{R}$. So finally we will have a classification of *all* local homogeneous Riemannian 3-manifolds.

For the 3 special cases $n^a = \delta^a_i$, $i = 1, \ldots, 3$ an explicit description of the Lorentzian 3-spaces of nonflat Bianchi type has been given in [15]. However a complete classification of all homogeneous Lorentzian 3-spaces need to control systematically the effect of different orientations (4.6). Presently, this problem still remains to be solved.

We proceed as follows: Sect. 4.2.1 derives and describes the classifying spaces (K^n, κ^n) of n-dimensional Lie algebras. An index function $J : K^n \to \mathbb{N}_0$ is related naturally to the topology κ^n. Using the index J, the topology κ^3 is described explicitly for both, the real and complex case. Sect. 4.2.2 relates the topology κ^n to the Zariski topology, and explains, via Lie algebra cohomology, why semisimple Lie algebras, and more generally all rigid ones, do not admit deformations in the category given by K^n. Sec. 4.2.3 then reviews the classification of all local homogeneous Riemannian 3-manifolds according to the algebraic structure of the Ricci curvature, and gives a comparison with the partial results for the 3 already mentioned cases of Lorentzian signature from [15].

4.2.1 Classifying spaces of local isometries

Since the local isometry subgroups of a local homogeneous manifold are Lie groups, it is useful, before trying to classify the homogeneous manifolds, to find first all possible contractions and, more generally, all possible limit transitions between real (or complex) Lie algebras of fixed dimension, and to uncover the natural topological structure of the space of all such Lie algebras. The topology is given by the algebraic properties of the Lie algebras. The space W^n of all structure constants of real n-dimensional Lie algebras carries the subspace topology induced from the Euclidean $\mathbb{R}^{n^3}$ (see [49]). The quotient topology κ^n, obtained from this topology w.r.t. equivalence by GL(n) isomorphisms, renders the space K^n of all n-dimensional Lie algebras into a T_0 topological space, which is not T_1 for $n \geq 2$. This non-T_1 topology has also been described in [50]. In Sect. 3 below, we derive it with some new method, employing an index function on the algebra. This approach is somehow inspired by Morse theory. For $n \geq 2$, the space K^n contains some non-closed point A, which has a special limit, to another point B. The inverse of such a transition from A to B is a deformation of the algebra of B into the algebra A. Note that, unlike in [20, 19], here transitions are defined to include also trivial constant limits. This has the advantage that also a trivial contraction (e.g. in the sense of Inönü-Wigner) is a transition. This definition is fully compatible with a partial order $A \geq B$, which is taken to be the specialization order already used in [20, 19]. This choice of partial order is naturally related to a Morse like potential J, decomposing K^n into subsets of different level.

A (real) *(finite-dimensional) Lie algebra* is a (real) vector space V of dimension n, equipped with a skew symmetric bilinear product $[\cdot, \cdot]$, satisfying the *Jacobi condition* $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$. The evaluation of the Lie bracket

$[\cdot,\cdot]$ on a complete set of basis vectors $\{e_i\}_{i=1,\ldots,n}$ yields a description of the Lie algebra by a set of structure constants $\{C^k_{ij}\}_{i,j,k=1,\ldots,n}$ from Eq. (4.5). Equivalently the endomorphisms $C_i := \mathrm{ad}(e_i)$, $i = 1,\ldots,n$, from the adjoint representation $\mathrm{ad} : e_i \to [e_i,\cdot]$, carry the same information on the algebra. Note that this description is overcomplete: Due to its antisymmetry, the Lie algebra is already completely described by the $(n-1)\times(n-1)$-matrices $C_{<i>}$, $i = 1,\ldots,n$, each with components C^k_{ij} $j,k = 1,\ldots,n-1$. But, as we will see, also this description may still carry redundancies.

The bracket $[\cdot,\cdot]$ defines a Lie algebra, iff the structure constants satisfy the $n\{\binom{n}{2}+\binom{n}{1}\}$ antisymmetry conditions

$$C^k_{[ij]} = 0, \tag{4.7}$$

and, corresponding to the Jacobi condition, the $n\cdot\binom{n}{3}$ quadratic compatibility constraints

$$C^l_{[ij}C^m_{k]l} = 0 \tag{4.8}$$

with nondegenerate antisymmetric indices i,j,k.

Here we only deal with finite-dimensional Lie algebras. Hence the adjoint representation in $\mathrm{End}(V)$ gives a natural associative matrix representation of the algebra, generated by the matrices C_i. Using this representation, the associativity of the matrix product $C_i\cdot C_j$ implies with $[C_i,C_j] = C^k_{ij}C_k$ that the Jacobi condition (4.8) is an *identity* following already from Eq. (4.7). However, if we do not use this extra knowledge from the adjoint representation, then, for $n>2$, Eq. (4.8) yields algebraic relations independent of Eq. (4.7).

For $n \geq 2$ there exists an irreducible tensor decomposition $C = D + V$, i.e.

$$C^k_{ij} = D^k_{ij} + V^k_{ij}, \tag{4.9}$$

where D is the tracefree part, i.e. $\mathrm{tr}(D_i) := D^k_{ik} = 0$, and V is the vector part,

$$V^k_{ij} := \delta^k_{[i}v_{j]}, \tag{4.10}$$

given by $v_i := \frac{2}{1-n}\mathrm{tr}(C_i)$, $i = 1,\ldots,n$. The Lie algebra is *tracefree* (corresponding Lie groups are *unimodular*) iff $V \equiv 0$, and it is said to be of *pure vector type* iff $D \equiv 0$. For each n, there exists exactly one non-Abelian pure vector type Lie algebra, denoted by V^n. For $n = 3$, the latter is the Bianchi type V , and the decomposition (4.9) is given by

$$C^k_{ij} = \varepsilon_{ijl}(n^{lk} + \varepsilon^{lkm}a_m), \quad D^k_{ij} = \varepsilon_{ijl}n^{lk}, \quad v_i = 2a_i, \tag{4.11}$$

where n^{ij} is symmetric and ε^{ijk} is the usual antisymmetric tensor (cf. also [51]). Hence for $n = 3$, the Jacobi condition (4.8) can be written as

$$n^{lm}a_m = 0. \tag{4.12}$$

These 3 nontrivial relations are in general independent of Eq. (4.7).

For arbitrary n, the space of all sets $\{C^k_{ij}\}$ satisfying the Lie algebra conditions (4.7) and (4.8) is a subvariety $W^n \subset \mathbb{R}^{n^3}$, with a dimension

$$\dim W^n \leq n^3 - \frac{n^2(n+1)}{2} = \frac{n^2(n-1)}{2}, \tag{4.13}$$

bounded by Eq. (4.7). For $n \geq 3$ the inequality is strict, because (4.8) is non trivial in general. For $n = 3$, the bound (4.13) reads $\dim W^n \leq 9$, and taking into account the 3 additional relations of Eq. (4.12) actually yields $\dim W^n = 6$.

GL(n) basis transformations act on a given set of structure constants as GL(n) tensor transformations:

$$C_{ij}^k \to \tilde{C}_{ij}^k := (A^{-1})_h^k \; C_{fg}^h \; A_i^f \; A_j^g \quad \forall A \in \mathrm{GL}(n). \tag{4.14}$$

On W^n this yields a natural equivalence relation $C \sim \tilde{C}$, defined by

$$C_{ij}^k \sim \tilde{C}_{ij}^k :\Leftrightarrow \exists A \in \mathrm{GL}(n) : \tilde{C}_{ij}^k = (A^{-1})_h^k \; C_{fg}^h \; A_i^f \; A_j^g, \tag{4.15}$$

with associated projection π to the quotient space,

$$\pi : \begin{cases} W^n & \to & K^n := W^n/\mathrm{GL}(n) \\ C & \mapsto & [C] \end{cases} \tag{4.16}$$

$$\dim W^n > \dim K^n \geq \dim W^n - n^2. \tag{4.17}$$

The upper bound in Eq. (4.17) is a strict one, because multiples of $\mathbb{1} \in \mathrm{GL}(n)$ give rise to equivalent points of W^n. Note however that, while, for a given $C \in W^n$, certain transformations $A \in \mathrm{GL}(n)$ transform $C \mapsto \tilde{C} \neq C$, others keep $C = \tilde{C}$ invariant. The latter transformations constitute the automorphism group $\mathrm{Aut}(C) \subset \mathrm{GL}(n)$ of the adjoint representation associated with C. In general, the GL(n) action on W^n is not free, i.e. there exist points C with $\dim \mathrm{Aut}(C) > 0$. So, Eqs. (4.13) and (4.17) provide only very weak bounds on $\dim K^n$, which is still unknown for general n (in the complex case, a more sophisticated upper bound estimate has been given in [52]). Note also that, e.g. for $n = 3$, the lower bound is trivial, because $\dim W^3 - 3^2 = -3 < 0$. Actually $\dim \mathrm{Aut}(C) \geq 3$ for all $C \in W^3$. In general, let us define the *automorphic dimension* of W^n as

$$\dim_{\mathrm{Aut}}(W^n) := \min_{C \in W^n} \{\dim \mathrm{Aut}(C)\}. \tag{4.18}$$

For any $A \in K^n$, consider a 1-parameter family of neighbourhoods $U_\varepsilon(A) \subset H(A)$ within the Hausdorff connected component $H(A)$ of A. Let us define the *dimension of the infinitesimal Hausdorff connected neighbourhood* of A as

$$\dim H(A) := \lim_{\varepsilon \to 0} \dim U_\varepsilon(A) \tag{4.19}$$

Then,

$$\begin{aligned} \dim W^n &= \max_{C \in W^n} \{\dim \pi^{-1}([C]) + \dim H([C])\} \\ &\leq \max_{C \in W^n} \{\dim \pi^{-1}([C])\} + \max_{C \in W^n} \{\dim H([C])\} \\ &= n^2 - \min_{C \in W^n} \{\dim \mathrm{Aut}(C)\} + \dim K^n. \end{aligned} \tag{4.20}$$

Using (4.18), the lower bound of Eq. (4.17) can be sharpened yielding

$$\dim W^n > \dim K^n \geq \dim W^n - n^2 + \dim_{\mathrm{Aut}}(W^n). \tag{4.21}$$

Note that $\dim \mathrm{Aut}(C) = \dim \mathrm{Aut}([C])$ for any $C \in W^n$. So $\dim_{\mathrm{Aut}}(W^n)$ actually depends only on K^n, and

$$\dim_{\mathrm{Aut}}(W^n) = \min_{C \in W^n} \{\dim \mathrm{Aut}([C])\} = \min_{A \in K^n} \{\dim \mathrm{Aut}(A)\} =: \dim_{\mathrm{Aut}}(K^n) \tag{4.22}$$

is the automorphic dimension of K^n.

The space K^n of isomorphism classes of n-dimensional Lie algebras is naturally rendered a topological space (K^n, κ^n), where the quotient topology κ^n is generated by the projection π from the subspace topology on $W^n \subset \mathbb{R}^{n^3}$. In order to describe (K^n, κ^n), let us first recall the axioms of *separation* (German: *Trennung*; cf. e.g. [53]):
T_0: For each pair of different points there is an open set containing only one of both.
T_1: Each pair of different points has a pair of open neighborhoods with their intersection containing none of both points.
T_2 (*Hausdorff*): Each pair of different points has a pair of disjoint neighborhoods.
It holds: $T_2 \Rightarrow T_1 \Rightarrow T_0$. Often it is more convenient to use the equivalent characterization of the separation axioms in terms of sequences and their limits:
$T_0 \Leftrightarrow$ For each pair of points there is a sequence converging only to one of them.
$T_1 \Leftrightarrow$ Each constant sequence has at most one limit.
$T_2 \Leftrightarrow$ Each sequence, indexed by a directed partially ordered set, has at most one limit.
T_1 is equivalent to the requirement that each 1-point set is closed. Actually, for $n \geq 2$, the topology κ^n is not T_1, but only T_0. This means that there exists some point $A \in K^n$, which is not closed, or in other words, there is a non-trivial *transition* from A to $B \neq A$ in $\mathrm{cl}\{A\}$. Non-trivial ($A \neq B$) transitions are special limits, which exist only due to the non-T_1 property of κ^n. Here transitions from A to B are defined by

$$A \geq B :\Leftrightarrow B \in \mathrm{cl}\{A\}. \tag{4.23}$$

By this definition, transitions are transitive and yield a natural partial order. A transition $A \geq B$ is non-trivial, iff $A > B$.

In the following we want to construct a minimal graph for the classifying space (K^n, κ^n). Let us associate an *arrow* $A \to B$ to a pair of algebras $A, B \in K^n$, with $A > B$, such that there exists no $C \in K^n$ with $A > C > B$. We call A the *source* and B the *target* of the arrow $A \to B$. Now we define a discrete *index* function $J : K^n \to \mathbb{N}_0$ as following: We start with the unique minimal element I^n, to which we assign the minimal index $J(\mathrm{I}^n) = 0$. Then, for $i \in \mathbb{N}_0$, we assign the index $J(S) = i + 1$ to the source algebra S of any arrow pointing towards a target algebra T of index $J(T) = i$, until, eventually for some index $J = i_{\max}$ there is no arrow to any target algebra T with $J(T) = i_{\max}$. Let us denote the subsets of all elements with index i as *levels* $L(i) \subset K^n$.

For $n \geq 2$, K^n is directed towards its *minimal element*, the Abelian Lie algebra I^n, constituting its only *closed point*. For $n \geq 3$ there are points in K^n which are neither open nor closed.

Open points correspond to *locally rigid* Lie algebras C, i.e. those which cannot be deformed to some $A \geq C$ with index $J(A) > J(C)$. In this sense, the *open points* in K^n are its *locally maximal elements*.

Isolated open points correspond to *rigid* Lie algebras C, i.e. those which cannot be deformed to any $A \in K^n$ with $A \not\leq C$ and index $J(A) \geq J(C)$. In this sense, the *isolated* *open points* are the *locally isolated maximal elements*. In Sect. 4.2.2 the isolated open points are considered also from the topologically dual perspective.

K^1 contains only the Abelian algebra I^1. K^2 contains 2 algebras, the Abelian I^2 and the isolated open point V^2, with a non-trivial transition $\mathrm{V}^2 \to \mathrm{I}^n$. The structure of K^3 is depicted explicitly in Fig. 1. In [20, 19] also the (already much more involved) topological structure of K^4 is shown in similar detail.

Consider a 1-parameter set of matrices $A_t \in \mathrm{GL}(n)$ with $0 < t \leq 1$, having a well defined matrix limit

$$A_0 := \lim_{t \to 0} A_t \tag{4.24}$$

which is *singular*, i.e. $\det A_0 = 0$.

For given structure constants C^k_{ij} of a Lie algebra A let us define for $0 < t \leq 1$ further structure constants

$$C^k_{ij}(t) := (A_t^{-1})^k_h \; C^h_{fg} \; (A_t)^f_i \; (A_t)^g_j, \tag{4.25}$$

which, according to (4.15), all describe the same Lie algebra A.

If there is a well defined limit $C^k_{ij}(0) := \lim_{t \to 0} C^k_{ij}(t)$, which satisfies conditions (4.7) and (4.8), yielding well defined structure constants of a Lie algebra B, then the associated transition $A \leq B$ is called a *contraction*.

Moreover a contraction is called *Inönü-Wigner contraction* if there is a basis $\{e_i\}$ in which

$$A(t) = \begin{pmatrix} E_m & 0 \\ 0 & t \cdot E_{n-m} \end{pmatrix} \qquad \forall t \in [0, 1], \tag{4.26}$$

where E_k denotes the k-dimensional unit matrix (cf. [54] and [57]). Given the decomposition (4.26), it was shown in [54] that, the limit $C^k_{ij}(0)$ exists iff $e_i, i = 1, \ldots, m$ span a subalgebra W of A, which then characterizes the contraction.

The elements of K^3 are well known to correspond to the famous Bianchi Lie algebras, classified independently by Lie [58] and Bianchi [59]. For all types of Bianchi Lie algebras I up to IX an explicit description can be given in terms of the nonvanishing matrices $C_{<i>}$, $i = 1, \ldots, 3$, of some adjoint representation. This representation can be normalized modulo an overall scale of the basis e_1, e_2, e_3, and moreover C_3 can be chosen in some normal form ([20, 19] use the Jordan normal form).

In the *semisimple* representation category, there are only the simple Lie algebras $\mathrm{VIII} \equiv \mathrm{so}(1,2) = \mathrm{su}(1,1)$ and $\mathrm{IX} \equiv so(3) = su(2)$. All other algebras are in the *solvable* representation category. They all have an Abelian ideal $\mathrm{span}\{e_1, e_2\}$. Hence, with vanishing $C_{<1>} = C_{<2>} = 0$, they are all characterized by $C_{<3>}$ only.

Transitions then arise both by *algebraic* and *geometric specialization* of the normal form of $C_{<3>}$, i.e. by degenerating of its eigenvalues or increasing multiplicities of its eigenvectors.

Anlong any line, specialization terminates at the unique Abelian Lie algebra I.

Fig. 1 shows on each horizontal level the algebras of equal index, which are the sources for the level below, and possible targets for the level above.

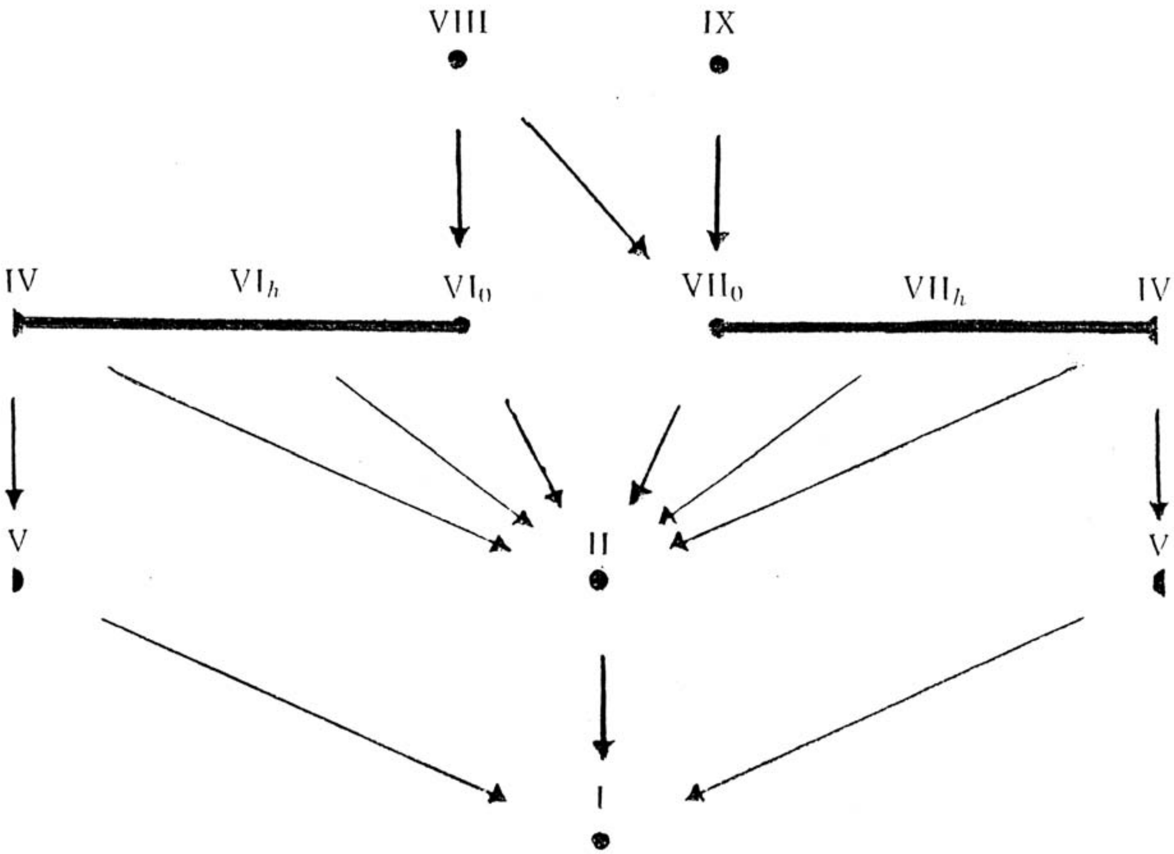

Figure 1: The topological space K^3 (right and left images have to be identified for the algebras IV and V; the locally maximal algebras IV, VI$_h$ and VII$_h$, $0 \leq h < \infty$, form a 1-parameter set of sources of arrows).

Fig. 2 gives the analogous picture for the space $K^3_{\mathbb{C}}$ of 3-dimensional complex Lie algebras.

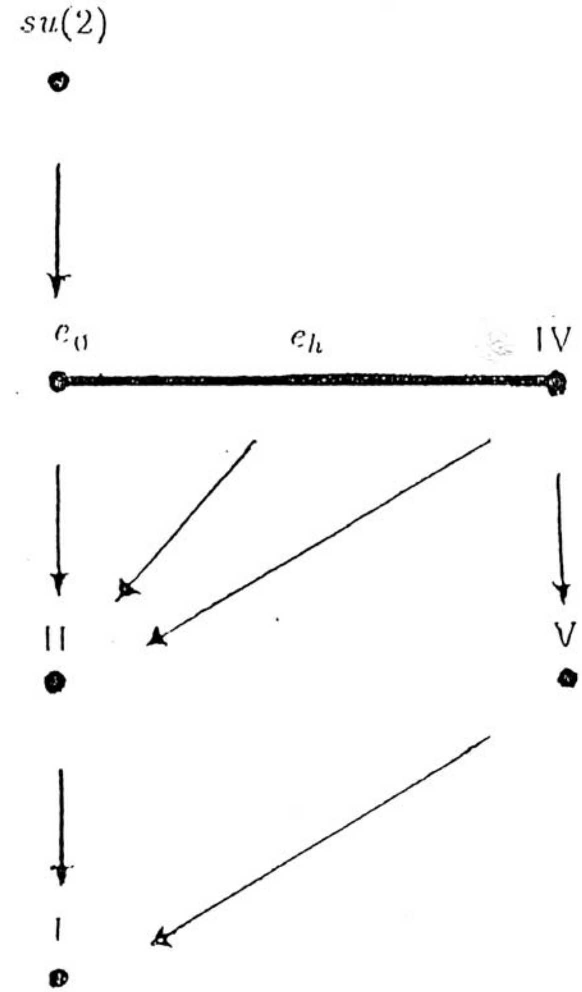

Figure 2: The topological space $K^3_{\mathbb{C}}$ (the locally maximal algebras IV, e_h, $0 \leq h < \infty$, form a 1-parameter set of sources of arrows).

4.2.2 Zariski dual topology and Lie algebra cohomology

Now note that for any topology there exists a *dual topology* by exchanging open and closed sets. Applied to the topology κ^n, open points of K^n become closed and closed points become open for the dual topology. Furthermore source and target of arrows interchange in the dual topology, i.e. their arrows change their direction. In the dual topology the rigid Lie algebras correspond to isolated closed points. Actually all semisimple Lie algebras are such isolated closed points.

Recall now, that on any algebraic variety there is a unique topology, called the *Zariski topology*, such that its closed subsets correspond to algebraic subvarieties. In this sense the topology κ^n turns naturally out to be the dual of the Zariski topology on K^n. So, what is the meaning of this *Zariski dual topology* of K^n and, more specifically of its *closed points* and *isolated closed points* ? To answer that question, note first that the reversed arrows of the dual topology correspond to some "inverse limit" of the transitions along them. More generally the dual of any transition might be called *spontaneous deformation* in K^n. This has to be distinguished from a *parametrical deformation* in K^n, which is given by a continuous change of parameters within a Hausdorff connected component of K^n. In the Zariski dual topology of K^n, the closed points cannot be source of spontaneous deformations in K^n, and the isolated closed points admit neither spontaneous nor parametrical deformations.

Actually deformations of Lie algebras can also be considered from a slightly different point of view, using *Lie algebra cohomology*, introduced in [60] and [61]. Let us consider a cochain complex

$$0 \rightarrow C^0 \overset{\delta_0}{\rightarrow} C^1 \overset{\delta_1}{\rightarrow} C^2 \rightarrow \dots \ . \tag{4.27}$$

Its cochain spaces

$$C^k(A,\mathcal{R}) := \{f : \overbrace{A \otimes \cdots \otimes A}^{k} \rightarrow \mathcal{R} | f \text{ linear, antisymmetric}\} \tag{4.28}$$

can generally be defined for any A-module $\mathcal{R}$ over some Lie algebra A. The coboundary operators are given by

$$\begin{aligned} \delta_k f(x_1,\dots,x_n) \ &:= \ \sum_{i=1}^{k}(-1)^{k+i} x_i f(x_1,\dots,\hat{x}_i,\dots,x_n) \\ &+\sum_{i,j=1}^{k}(-1)^{i+j} f(x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_n,[x_i,x_j]). \end{aligned} \tag{4.29}$$

f is an *k-cocycle* iff $\delta_k f = 0$. $Z^k(A,\mathcal{R}) := \ker\delta_k$. f is an *k-coboundary* iff $f = \delta_{k-1}g$. $B^k(A,\mathcal{R}) := \text{im}\delta_{k-1}$. The coboundary (4.29) satisfies $\delta^2 = 0$, hence $B^n(A,\mathcal{R}) \subset Z^n(A,\mathcal{R})$ and the k^{th} *cohomology* is defined as $H^k(A,\mathcal{R}) := Z^k(A,\mathcal{R})/B^k(A,\mathcal{R})$.

Let us restrict now for simplicity to complexes (4.27) with $\mathcal{R} = A$, where the left multiplication by x is just given by the adjoint action $\text{ad}_x := [x,\cdot]$, and write $C^k \equiv C^k(A,A)$ and $H^k \equiv H^k(A,A)$, keeping in mind that these quantities all depend on the algebra $A \in K^n$.

The *deformation* $[\cdot,\cdot]_\varepsilon$ of the product $[\cdot,\cdot]$ of some algebra A, can be written as a formal power series

$$[x,y]_\varepsilon = [x,y] + \varepsilon F_1(x,y) + \varepsilon^2 F_2(x,y) + \dots. \tag{4.30}$$

If the product $[\cdot,\cdot]$ is defined in some category, e.g. the category Lie products W^n of Lie algebras in K^n, the formally deformed product $[\cdot,\cdot]_\varepsilon$ is in general not well defined on the same category, but only on some extended category. In order to be still a product in the same category, here in W^n, the deformed product $[\cdot,\cdot]_\varepsilon$ has to satisfy an infinite number of *deformation equations*, namely for all $k \in \mathbb{N}_0$ the coefficients of the formal power series have to satisfy

$$\sum_{\substack{x,y,z\\ \text{cyclic}}} \sum_{i+j=k} F_i(F_j(x,y),z) + F_j(F_i(x,y),z) = 0, \tag{4.31}$$

with $F_0 \equiv [\cdot,\cdot]$. For Lie products in W^n, the equation for $k=0$ corresponds to the Jacobi condition, and the *infinitesimal deformation equation*, i.e. the equation for $k=1$, can be expressed with $\delta \equiv \delta_2$ from (4.29) as

$$\delta F_1 = 0. \tag{4.32}$$

So we see that Z^2 is the set of *infinitesimal deformations* of elements of W^n. Some of these deformations yield the again the original algebra $A \in K^n$, i.e. they are deformations along the GL(n)-orbit through $F_0 \in W^n$. These *trivial infinitesimal deformations* are elements of B^2. Hence H^2 contains just the *non-trivial infinitesimal deformations*. If some algebra $A \in K^n$ satisfies $H^2 \equiv H^2(A,A) = 0$, then this algebra cannot be source of *infinitesimal deformations*. The latter may, corresponding to the definitions above, be divided into *infinitesimal spontaneous deformations* and *infinitesimal parametrical deformations*, where the former are the duals of transitions and the latter generate parametrical deformations within the Hausdorff connected component of A. So, if $A \in K^n$ is an isolated open point w.r.t. the original topology of K^n or, equivalently, an isolated closed point w.r.t. the Zariski dual topology of K^n, then there are no non-trivial infinitesimal deformations of the product $F_0 \in W^n$; rather all infinitesimal deformations are within $\pi^{-1}(A) \subset W^n$. For these algebras $H^2 = 0$. In particular, the latter is known to be true for all semisimple Lie algebras.

4.2.3 Classifying spaces of local homogeneous geometries

Now we will construct a classifying space for local homogeneous Riemannian 3-manifolds.

The KS spaces appear as a limit of Bianchi IX spaces, in which the Bianchi IX isometry is still maintained, but no longer transitive. Hence it is sufficient to consider local 3-manifolds of Bianchi Lie isometry.

For the present case of Riemannian 3-spaces g_{ab} has a definite sign. Let us consider the 3-geometry *modulo* a transformation of the its global sign,

$$g_{ab} \to -g_{ab}. \tag{4.33}$$

Then we can normalize the global sign with $\det(g_{ab}) > 0$ to $\epsilon_1 = \epsilon_3 = \epsilon_3 = 1$ in Eq. (4.3).

The choice of real parameters s, t, w of Eq. (4.3) simplifies calculations in a specific triad basis of the Lie algebra corresponding to the isometry of the geometry. This basis is chosen in consistency with the representations of [51] and [62]. It can be represented by matrices (e^a_α), with anholonomic $a = 1, 2, 3$ of the generators of the algebra, and holonomic coordinate columns $\alpha = 1, 2, 3$.

The structure constants can be reobtained from ds^2 and the triad by

$$C_{ijk} = ds^2([e_i, e_j], e_k), \qquad C^k_{ij} = C_{ijr} g^{rk}. \tag{4.34}$$

The metrical connection coefficients are determined as

$$\Gamma^k_{ij} = \frac{1}{2} g^{kr}(C_{ijr} + C_{jri} + C_{irj}). \tag{4.35}$$

The Ricci tensor is

$$R_{ij} := R^k_{ikj} = \Gamma^f_{ij}\Gamma^e_{fe} - \Gamma^f_{ie}\Gamma^e_{fj} + \Gamma^e_{if}C^f_{ej}. \tag{4.36}$$

From (4.36) we may form the following scalar invariants of the geometry: The Ricci curvature scalar

$$R := R^i{}_i, \tag{4.37}$$

the sum of the squared eigenvalues

$$N := R^i{}_j R^j{}_i, \tag{4.38}$$

the trace-free scalar

$$S := S^i{}_j S^j{}_k S^k{}_i = R^i{}_j R^j{}_k R^k{}_i - RN + \frac{2}{9}R^3, \tag{4.39}$$

where $S^i{}_j := R^i{}_j - \frac{1}{3}\delta^i{}_j R$, and, related to the York tensor,

$$Y := R_{ik;j} g^{il} g^{jm} g^{kn} R_{lm;n}, \qquad \text{with} \tag{4.40}$$

$$\begin{aligned} R_{ij;k} &:= e^\alpha_i e^\beta_j e^\gamma_k R_{\alpha\beta;\gamma} \\ &= e^l_{\alpha;\beta} e^\alpha_m e^\beta_k (\delta^m_i \delta^n_j + \delta^n_i \delta^m_j) R_{ln}. \end{aligned}$$

The 4 scalar invariants above characterize a local homogeneous Riemannian 3-space.

It is $N = 0$, iff the Riemannian 3-space is the unique flat one. This has a transitive isometry of Bianchi type I, and also admits the left-invariant (but not transitive) action of the Bianchi group VII_0 on its 2-dimensional hyperplanes (cf. [63, 64]).

In the following we take the flat Riemannian 3-space as a center of projection for the non-flat Bianchi or KS geometries. These satisfy $N \neq 0$. The invariant N then parametrizes (like e^{-2s}) the homogeneous conformal scale on the 3-manifold under consideration. A homogeneous conformal, i.e. homothetic, rescaling of the metric,

$$g_{ij} \to \sqrt{N} g_{ij}, \tag{4.41}$$

yields the following normalized invariants, which depend only on the homogeneously conformal class of the geometry:

$$\hat{N} := 1, \quad \hat{R} := R/\sqrt{N}, \quad \hat{S} := S/N^{3/2}, \quad \hat{Y} := Y/N^{3/2}. \tag{4.42}$$

For a non-flat Riemannian space, the invariant $\hat{Y}$ vanishes, iff the 3-geometry is conformally flat. Note that a general conformal transformation is not necessarily homogeneous. Hence, there may exist homogeneous spaces, which are in the same conformal class, but in different homogeneous-conformal classes.

Note that, in the Riemannian case, under (4.33), N is invariant, while $\hat{R}$, $\hat{S}$ and $\hat{Y}$ just all reverse their sign. Furthermore, a rescaling (4.41) does not change the Bianchi or KS type of isometry.

In the Lorentzian case, N may change its sign, and also it may be zero even in the non-flat case. Nevertheless, for non-flat spaces with $N \neq 0$ the invariants (4.42) can be defined even in the Lorentzian case.

So we can now concentrate on the classifying space of non-flat local homogeneous Riemannian 3-geometries *modulo* the global sign (4.33) and *modulo* homogeneous conformal transformations (4.41) for each fixed Bianchi type. This *moduli space* can be parametrized by the invariants $\hat{R}$, $\hat{S}$ and $\hat{Y}$, given for each fixed Bianchi type as a function of the anisotropy parameters t and w.

A minimal cube, in which the classifying moduli space can be imbedded, is spanned by $\hat{R}/\sqrt{3}, \sqrt{6}\hat{S} \in [-1, 1]$ and $2 \tanh \hat{Y} \in [0, 2]$.

Below, Fig. 3 describes those points of the moduli space which are of Bianchi types VI/VII or lower level, Fig. 4 likewise points of Bianchi types VIII/IX.

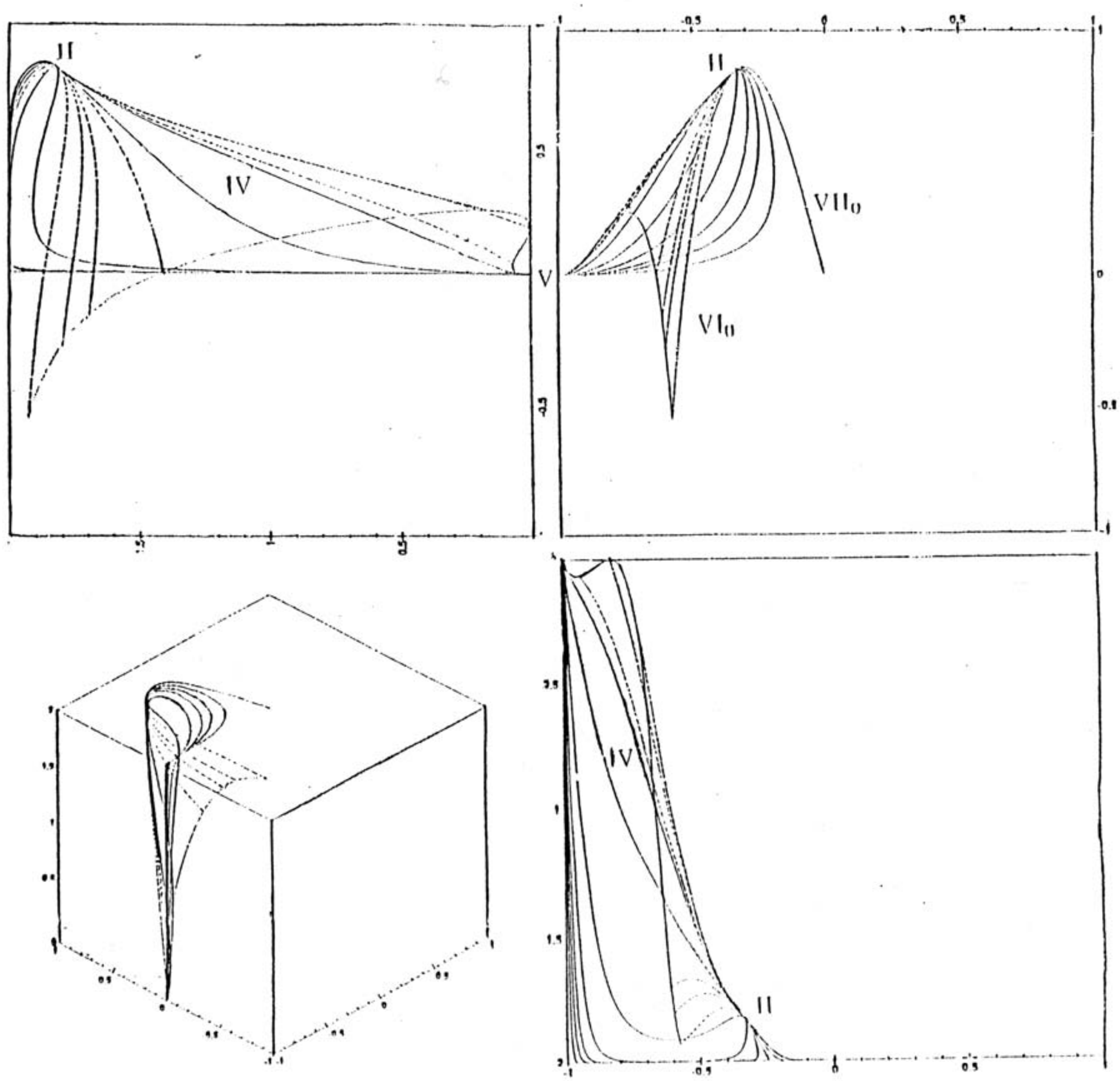

Figure 3: Riemannian Bianchi geometries II, IV, V, VI$_h(w=0)$, VI$_h$ ($\sqrt{h}=0,\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{5}{8},1,2$), VII$_h$ ($\sqrt{h}=0,\frac{1}{7},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{1}{2},1$); w.r.t. the common origin, the axes of the 3 planar diagrams, are: $\hat{R}/\sqrt{3}$ to the right, $\sqrt{6}\hat{S}$ up, and $2\tanh\hat{Y}$ both, left and down.

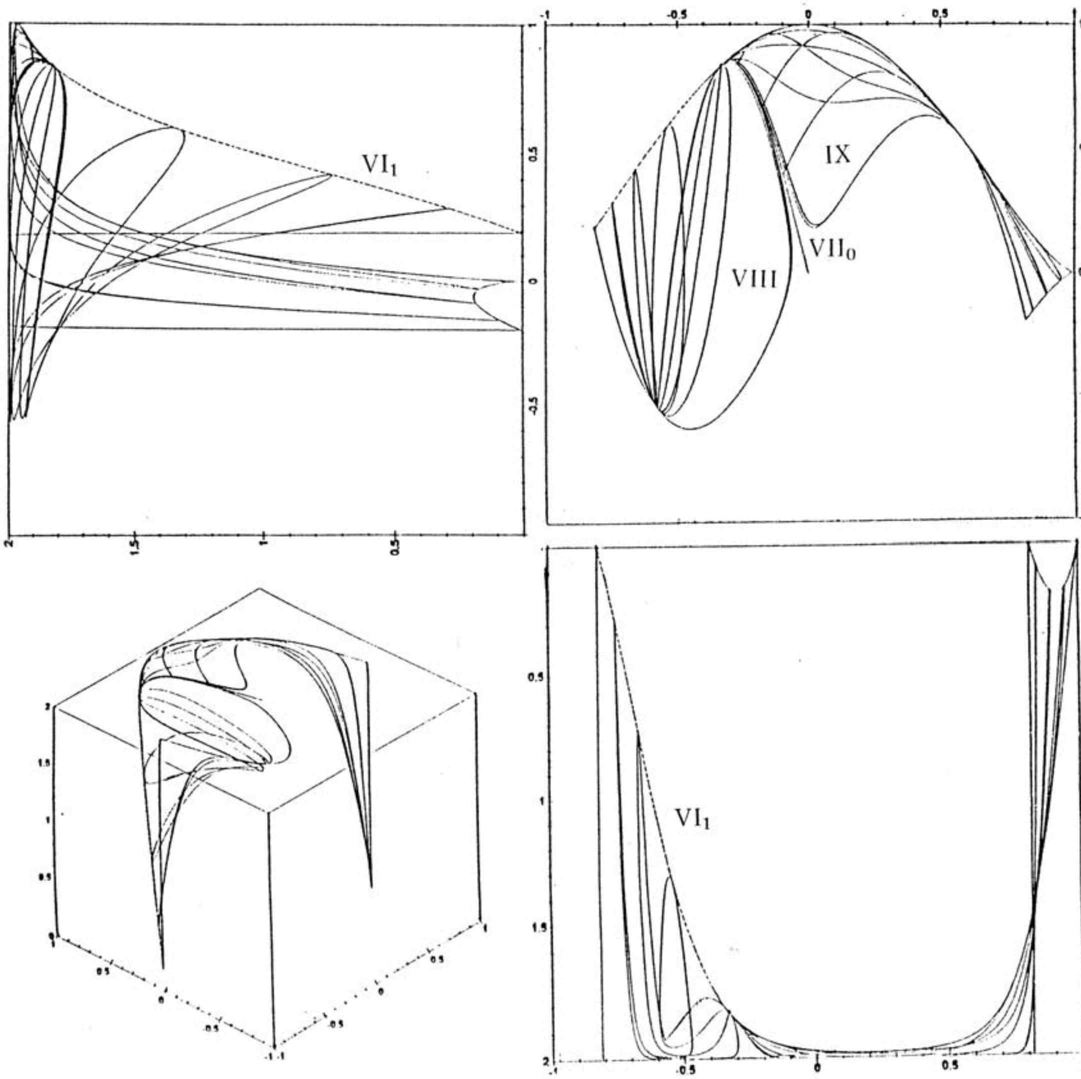

Figure 4: Riemannian Bianchi geometries II, V, VI_0, VI_1, VII_0, $\mathrm{VIII}(t,w)$ $(t=-5,-1,0,1,5)$, $\mathrm{IX}(t,w)$ $(t=0,\frac{1}{2},1,2,5)$; w.r.t. the common origin, the axes of the 3 planar diagrams, are: $\hat{R}/\sqrt{3}$ to the right, $\sqrt{6}\hat{S}$ up, and $2\tanh\hat{Y}$ both, left and down.

For a homogeneous space with 2 equal Ricci eigenvalues the corresponding point in the $\hat{R}$-$\hat{S}$-plane lies on a double line L_2, which has a range defined by $|\hat{R}| \leq \sqrt{3}$ and satisfies the algebraic equation

$$162\hat{S}^2 = (3 - \hat{R}^2)^3 \; . \tag{4.43}$$

All other algebraically possible points of the $\hat{R}$-$\hat{S}$-plane lie inside the region surrounded by the line L_2. At the branch points $\hat{R} = \pm\sqrt{3}$ of L_2 all Ricci eigenvalues are equal. These homogeneous spaces possess a 6-dimensional isometry group. Homogeneous spaces possessing a 4-dimensional isometry group are represented by points on L_2.

If one Ricci eigenvalue equals R, i.e. if there exists a pair $(a, -a)$ of Ricci eigenvalues, the corresponding point in the $\hat{R}$-$\hat{S}$-plane lies on a line L_{+-}, defined by the range $|\hat{R}| \leq 1$ and the algebraic equation

$$\hat{S} = \frac{11}{9}\hat{R}^3 - \hat{R}. \tag{4.44}$$

In the case that one eigenvalue of the Ricci tensor is zero, the corresponding point in the $\hat{R}$-$\hat{S}$-plane lies on a line L_0, defined by the range $|\hat{R}| \leq \sqrt{2}$ and the algebraic equation

$$\hat{S} = \frac{\hat{R}}{2}(1 - \frac{5}{9}\hat{R}^2). \tag{4.45}$$

For Eqs. (4.43),(4.44),(4.45) see also [18].

At the branch points of the curve L_2 the Ricci tensor has a triple eigenvalue, which is negative for geometries of Bianchi type V, and positive for type IX geometries with parameters $(t, w) = (0, 0)$. These constant curvature geometries are all conformally flat with $\hat{Y} = 0$. Besides the flat Bianchi I geometry, the remaining conformally flat spaces with $\hat{Y}$ are the KS space $(\hat{R}, \hat{S}, \hat{Y}) = (\sqrt{2}, -\frac{\sqrt{2}}{18}, 0)$ and, point reflected, the Bianchi type III_c, corresponding to the initial point of a Bianchi III line segment ending at the Bianchi II point in Fig. 3.

The point $(-1, 0, 0)$ of Fig. 3 admits both types, Bianchi V and VII_h with $h > 0$. Nevertheless, this point corresponds only to one homogeneous space, namely the space of constant negative curvature. This is possible, because this space has a 6-dimensional Lie group, which contains the Bianchi V and VII_h subgroups. Note that in the flat limit $\mathrm{V} \to \mathrm{I}$, the additional Bianchi groups VII_h change with $h \to 0$.

Similarly, the Bianchi III points of Fig. 3 lie on the curve L_2 of the $\hat{R} - \hat{S}$ diagram. However, these points are also of Bianchi type VIII. In fact, each of them correspond to one homogeneous geometry only. However, the latter admits a 4-dimensional isometry group, which has two 3-dimensional subgroups, namely Bianchi III and VIII, both containing the same 2-dimensional non-Abelian subgroup.

Altogether, the location of Riemannian Bianchi (and KS) spaces is consistent with the topology κ^3 of the space of Bianchi Lie algebras.

Our classifying moduli space of local homogeneous Riemannian 3-spaces is a T_2 (Hausdorff) space. But it is not a topological manifold: The line of VII_0 moduli is a common boundary of 3 different 2-faces, namely that of the IX moduli, that of the VIII moduli, and with $h \to 0$ that of all moduli of type VII_h with $h > 0$. Like the moduli space, also the full classifying space is not locally Euclidean; rather both are stratifiable varieties.

The general admissibility of certain isometry groups of 3-dimensional Riemannian and Lorentzian metrics has been investigated in [65] and [66] respectively.

In the following we proceed vice versa: We calculate the characteristic scalar invariants (4.42) for non-flat Bianchi geometries, in different signature cases. Because here the dimension is odd, the transformation $g_{ab} \to -g_{ab}$ changes the sign of $\det(g_{ab})$. Furthermore, it keeps N invariant and maps $R \to -R$, $S \to -S$ and $Y \to -Y$.

Therefore let us normalize this global sign by $\det(g_{ab}) > 0$ for any g_{ab}, Riemannian or Lorentzian. Then for the latter case there remain 3 subcases of signature $\epsilon := (\epsilon_1, \epsilon_2, \epsilon_3)$ to be studied: $\epsilon = (+--)$, $(-+-)$, $(--+)$.

For Fig. 5 and 6 below, we consider for different signatures the rescaled cube spanned by the scalar invariants $\hat{R}/\sqrt{3}, \sqrt{6}\hat{S} \in [-1,1]$ and $2\tanh\hat{Y} \in [0,2]$.

The scalar invariants for all non-flat Riemannian Bianchi geometries are as follows:

$$\begin{aligned}
\hat{R}_{\mathrm{II}} &= -\frac{\sqrt{3}}{3} \\
\hat{S}_{\mathrm{II}} &= \frac{16\sqrt{3}}{81} \\
\hat{Y}_{\mathrm{II}} &= \frac{8\sqrt{3}}{9}
\end{aligned} \tag{4.46}$$

$$\begin{aligned}
\hat{R}_{\mathrm{IV}} &= -\frac{12\,e^{w}+1}{\sqrt{48\,e^{2\,w}+16\,e^{w}+3}} \\
\hat{S}_{\mathrm{IV}} &= \frac{16+72\,e^{w}}{9\,\left(48\,e^{2\,w}+16\,e^{w}+3\right)^{3/2}} \\
\hat{Y}_{\mathrm{IV}} &= \frac{8+8\,e^{w}+32\,e^{2\,w}}{\left(48\,e^{2\,w}+16\,e^{w}+3\right)^{3/2}}
\end{aligned} \tag{4.47}$$

$$\begin{aligned}
\hat{R}_{\mathrm{V}} &= -\sqrt{3} \\
\hat{S}_{\mathrm{V}} &= 0 \\
\hat{Y}_{\mathrm{V}} &= 0
\end{aligned} \tag{4.48}$$

In the next formulas, D represents an expression to simplify the following equations.

$$\begin{aligned}
D_{\mathrm{VI}_h} &:= 3+4\,(4\,h+1)\,e^{w}+2\,\left(1+16\,h+24\,h^2\right)e^{2\,w} \\
&\quad +4\,(4\,h+1)\,e^{3\,w}+3\,e^{4\,w} \\
\hat{R}_{\mathrm{VI}_h} &= -(D_{\mathrm{VI}_h})^{-\frac{1}{2}}\left(1+2\,(1+6\,h)\,e^{w}+e^{2\,w}\right) \\
\hat{S}_{\mathrm{VI}_h} &= \frac{8}{9}\,(D_{\mathrm{VI}_h})^{-\frac{3}{2}}\,(e^{w}+1)^4\left(2+(9\,h-5)\,e^{w}+2\,e^{2\,w}\right) \\
\hat{Y}_{\mathrm{VI}_h} &= 8\,(D_{\mathrm{VI}_h})^{-\frac{3}{2}}\,(e^{w}+1)^2\left(e^{4\,w}+(h-1)\,e^{3\,w}\right. \\
&\quad \left.+2\,\left(2\,h^2-5\,h+2\right)e^{2\,w}+(h-1)\,e^{w}+1\right)
\end{aligned} \tag{4.49}$$

$$\begin{aligned}
D_{\mathrm{VII}_h} &:= 3+4\,(4\,h-1)\,e^{w}+2\,\left(1-16\,h+24\,h^2\right)e^{2\,w} \\
&\quad +4\,(4\,h-1)\,e^{3\,w}+3\,e^{4\,w}
\end{aligned}$$

$$
\begin{aligned}
\hat{R}_{\mathrm{VII}_h} &= -(D_{\mathrm{VII}_h})^{-\frac{1}{2}}\left(1+2\,(6\,h-1)\,e^{w}+e^{2\,w}\right)\\
\hat{S}_{\mathrm{VII}_h} &= \frac{8}{9}\,(D_{\mathrm{VII}_h})^{-\frac{3}{2}}\,(e^{w}-1)^{4}\left(2+(9\,h+5)\,e^{w}+2\,e^{2\,w}\right)\\
\hat{Y}_{\mathrm{VII}_h} &= 8\,(D_{\mathrm{VII}_h})^{-\frac{3}{2}}\,(e^{w}-1)^{2}\left(e^{4\,w}+(h+1)\,e^{3\,w}\right.\\
&\quad \left.+2\,\left(2\,h^{2}+5\,h+2\right)e^{2\,w}+(h+1)\,e^{w}+1\right)
\end{aligned}
\tag{4.50}
$$

$$
\begin{aligned}
D_{\mathrm{VIII}} &:= 2\,e^{-2\,w+2\,t}+3\,e^{-2\,w+4\,t}+4\,e^{-2\,w+t}-4\,e^{t}+4\,e^{w}+3\,e^{2\,w}\\
&\quad +4\,e^{-w}+3\,e^{-2\,w}-4\,e^{-w+2\,t}+4\,e^{-2\,w+3\,t}+2+4\,e^{-w+t}\\
&\quad -4\,e^{w+t}-4\,e^{-w+3\,t}+2\,e^{2\,t}\\
\hat{R}_{\mathrm{VIII}} &= -(D_{\mathrm{VIII}})^{-\frac{1}{2}}(2\,e^{-w+t}+e^{-w}+e^{-w+2\,t}+e^{w}+2-2\,e^{t})\\
\hat{S}_{\mathrm{VIII}} &= -\frac{8}{9}\,(D_{\mathrm{VIII}})^{-\frac{3}{2}}\left(6\,e^{w+2\,t}+3\,e^{t+2\,w}+3\,e^{5\,t-2\,w}+14\,e^{-3\,w+3\,t}\right.\\
&\quad +6\,e^{-3\,w+2\,t}+6\,e^{-3\,w+4\,t}-3\,e^{-3\,w+t}+6\,e^{-w+4\,t}-14\,e^{3\,t}\\
&\quad -3\,e^{-3\,w+5\,t}-2\,e^{3\,w}-2\,e^{-3\,w}-15\,e^{-w+t}-3\,e^{-2\,w}\\
&\quad +15\,e^{-2\,w+3\,t}+18\,e^{w+t}-15\,e^{-w+3\,t}-15\,e^{2\,t}-3\,e^{2\,w}\\
&\quad +15\,e^{t}+18\,e^{-2\,w+4\,t}-18\,e^{-2\,w+t}+6\,e^{-w}-42\,e^{-w+2\,t}\\
&\quad \left.+6\,e^{w}-15\,e^{-2\,w+2\,t}-2\,e^{-3\,w+6\,t}+14\right)\\
\hat{Y}_{\mathrm{VIII}} &= 8\,(D_{\mathrm{VIII}})^{-\frac{3}{2}}\left(3\,e^{w+2\,t}-e^{t+2\,w}-e^{5\,t-2\,w}+6\,e^{-3\,w+3\,t}\right.\\
&\quad +3\,e^{-3\,w+2\,t}+3\,e^{-3\,w+4\,t}+e^{-3\,w+t}+3\,e^{-w+4\,t}-6\,e^{3\,t}\\
&\quad +e^{-3\,w+5\,t}+e^{3\,w}+e^{-3\,w}-6\,e^{-w+t}+e^{-2\,w}+6\,e^{-2\,w+3\,t}\\
&\quad +5\,e^{w+t}-6\,e^{-w+3\,t}-6\,e^{2\,t}+e^{2\,w}+6\,e^{t}+5\,e^{-2\,w+4\,t}-5\,e^{-2\,w+t}\\
&\quad \left.+3\,e^{-w}-18\,e^{-w+2\,t}+3\,e^{w}-6\,e^{-2\,w+2\,t}+e^{-3\,w+6\,t}+6\right)
\end{aligned}
\tag{4.51}
$$

$$
\begin{aligned}
D_{\mathrm{IX}} &:= 2+3\,e^{2\,w}+3\,e^{-2\,w}-4\,e^{-w}-4\,e^{-2\,w+t}+3\,e^{-2\,w+4\,t}+2\,e^{2\,t}\\
&\quad +2\,e^{-2\,w+2\,t}+4\,e^{-w+2\,t}-4\,e^{-2\,w+3\,t}-4\,e^{w}+4\,e^{t}\\
&\quad +4\,e^{-w+t}-4\,e^{-w+3\,t}-4\,e^{w+t}\\
\hat{R}_{\mathrm{IX}} &= (D_{\mathrm{IX}})^{-\frac{1}{2}}\left(-e^{w}-e^{-w+2\,t}-e^{-w}+2\,e^{-w+t}+2+2\,e^{t}\right)\\
\hat{S}_{\mathrm{IX}} &= -\frac{8}{9}\,(D_{\mathrm{IX}})^{-\frac{3}{2}}\left(6\,e^{w+2\,t}+3\,e^{t+2\,w}+3\,e^{5\,t-2\,w}-14\,e^{-3\,w+3\,t}\right.\\
&\quad +6\,e^{-3\,w+2\,t}+6\,e^{-3\,w+4\,t}+3\,e^{-3\,w+t}+6\,e^{-w+4\,t}-14\,e^{3\,t}\\
&\quad +3\,e^{-3\,w+5\,t}-2\,e^{3\,w}-2\,e^{-3\,w}+15\,e^{-w+t}+3\,e^{-2\,w}\\
&\quad +15\,e^{-2\,w+3\,t}-18\,e^{w+t}+15\,e^{-w+3\,t}+15\,e^{2\,t}+3\,e^{2\,w}\\
&\quad +15\,e^{t}-18\,e^{-2\,w+4\,t}-18\,e^{-2\,w+t}+6\,e^{-w}-42\,e^{-w+2\,t}\\
&\quad \left.+6\,e^{w}+15\,e^{-2\,w+2\,t}-2\,e^{-3\,w+6\,t}-14\right)\\
\hat{Y}_{\mathrm{IX}} &= -8\,(D_{\mathrm{IX}})^{-\frac{3}{2}}\left(-3\,e^{w+2\,t}+e^{t+2\,w}+e^{5\,t-2\,w}+6\,e^{-3\,w+3\,t}\right.\\
&\quad -3\,e^{-3\,w+2\,t}-3\,e^{-3\,w+4\,t}+e^{-3\,w+t}-3\,e^{-w+4\,t}+6\,e^{3\,t}\\
&\quad +e^{-3\,w+5\,t}-e^{3\,w}-e^{-3\,w}-6\,e^{-w+t}+e^{-2\,w}-6\,e^{-2\,w+3\,t}\\
&\quad +5\,e^{w+t}-6\,e^{-w+3\,t}-6\,e^{2\,t}+e^{2\,w}-6\,e^{t}+5\,e^{-2\,w+4\,t}+5\,e^{-2\,w+t}\\
&\quad \left.-3\,e^{-w}+18\,e^{-w+2\,t}-3\,e^{w}-6\,e^{-2\,w+2\,t}-e^{-3\,w+6\,t}+6\right)
\end{aligned}
\tag{4.52}
$$

In order to verify that, for given (R, N, S, Y) there exists only one Bianchi geometry, we fix N and examine the position of any Bianchi geometry in the cube of scalar invariants $(\hat{R}/\sqrt{3}, \sqrt{6}\hat{S}, 2\tanh\hat{Y})$.

For the pseudo-Riemannian homogeneous spaces, the analysis can be done quite analogously, but up to now, only partial results are known (see e.g. [63]). The conditions under which different signatures yield isometric spaces are not totally clear at present. A further problem is that here some of the invariants (4.42) might become singular for some non-flat spaces with $N = 0$.

Note also that the Lorentzian flat space accommodates not only the transitive Abelian Bianchi type I and the isometry group VII_0 of its space-like hyperplanes, but also further left-invariant Bianchi groups, which do not appear in the flat Riemannian case (cf. also [63]):

Naturally, one of them is the isometry group VI_0 of its Minkowskian hyperplanes. Further admissible Bianchi groups on the Minkowski space are the types II and V.

Actually Lorentzian Bianchi type II geometries are completely known: There exists a flat left-invariant Lorentz metric, and all non-flat ones are homothetically equivalent to each other. Non-flat Lorentzian Bianchi II geometries are in all 3 signature cases the same as the Riemannian ones, Eq. (4.46).

Bianchi geometries with signature $\epsilon = (+--)$ of type IV, V, VI_h and VII_h are the same as the Riemannian ones, Eqs. (4.47), (4.48), (4.49) and (4.49), respectively. Similarly, the analogous geometries of signature $\epsilon = (-+-)$ and $\epsilon = (--+)$ have both exactly the same values of $\hat{R}, \hat{S}, \hat{Y}$, although here these are different from Eqs. (4.47), (4.48), (4.49) and (4.49), respectively.

For type V this difference is just given by a reflection from constant negative curvature $\hat{R} = -\sqrt{3}$ to constant positive curvature $\hat{R} = \sqrt{3}$. Although for Bianchi type V the value of $\hat{R}$ changes with the signature ϵ, its property $\hat{Y} = 0$ is preserved for all 3 Lorentzian signatures as in the Riemannian case. Actually Lorentzian Bianchi V geometries are known to exist for any constant curvature, including the flat case [63], which however is excluded for our invariants (4.42).

According to the natural topology in the space K^3 of real Lie algebras, type IV interpolates between II and V, and both, VI_h and VII_h, converge to IV for $h \to \infty$. So IV, VI_h and VII_h have to behave under a change of signature consistently with the fixed point II and the reflection of V.

Finally Bianchi VIII and IX geometries are different for all signatures.

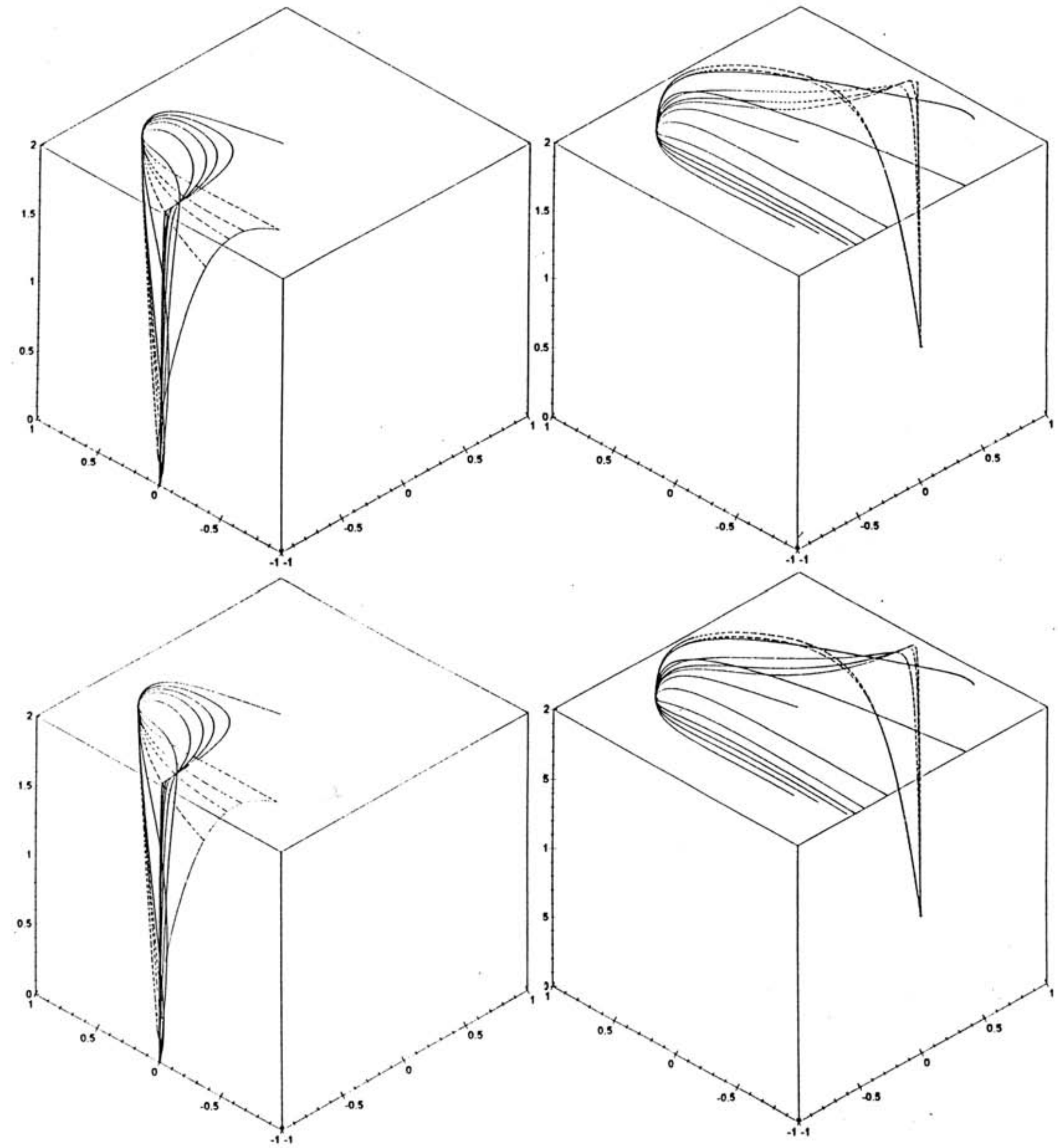

Figure 5: Bianchi types II, IV, V, VI_h, VII_h (h and w as in Fig. 3) for: $(+++)$ lower left, $(+--)$ top left, $(-+-)$ top right, $(--+)$ lower right; the axes are: $\hat{R}/\sqrt{3}$ to the right, $\sqrt{6}\hat{S}$ to the left, and $2\tanh\hat{Y}$ up.

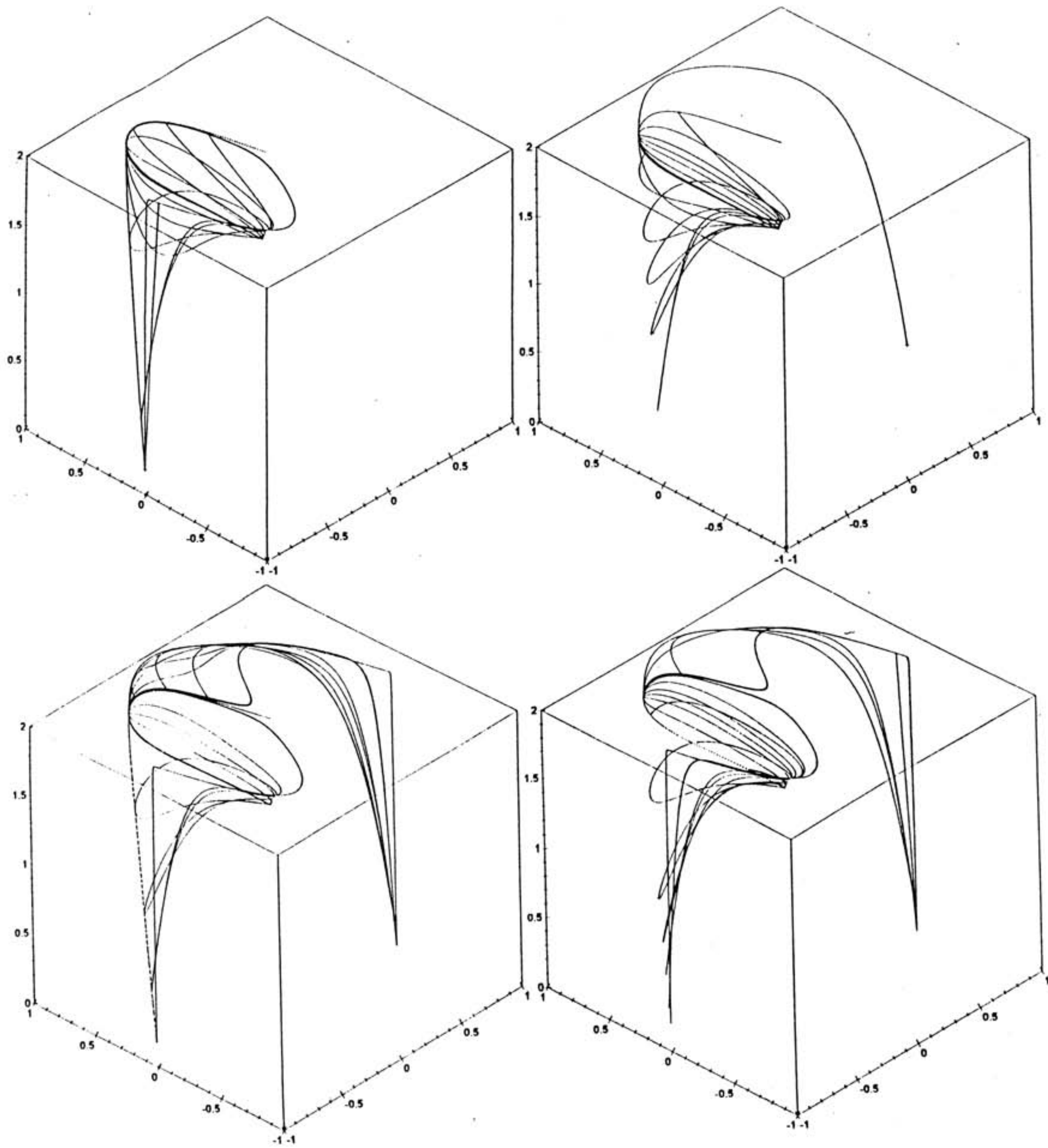

Figure 6: Bianchi types II, V, VI_0, VI_1, VII_0, VIII, IX (t, w as in Fig. 4) for: $(+++)$ lower left, $(+--)$ top left, $(-+-)$ top right, $(--+)$ lower right; the axes are: $\hat{R}/\sqrt{3}$ to the right, $\sqrt{6}\hat{S}$ to the left, and $2\tanh\hat{Y}$ up.

5. Cosmological application II: Rigidity of isometries

In this section we sketch the possible application of our results to isometries in mathematical cosmology. In particular, the stability of isometries may be an issue during the evolution of classical cosmological models or in minisuperspace models of quantum cosmology.

Previously obtained classifying spaces may be useful in the context of rigidity theorems for the variation of the local isometry type within a Riemannian or Lorentzian 4-geometry.

The traditionally considered $1+3$-dimensional inhomogeneous cosmological models with homogeneous Riemannian 3-hypersurfaces (see [18], [67], [68]), and a more general class of multidimensional geometries (see [16]), admit deformations between Riemannian Bianchi geometries. Such a deformation may also induce a change of the spatial anisotropy of the universe, which essentially affects physical quantities like its tunnelling rate [69]. Even for the more general multidimensional case, where $M = \mathbb{R} \times M_1 \times \ldots \times M_n$ with homogeneous M_1, our results provide an important piece of information, namely the complete control over the possible continuous deformations of the homogeneous external 3-space M_1.

Furthermore, the superspace of homogeneous Riemannian 3-geometries plays a key role for an understanding of the canonical quantization of a homogeneous universe. The homogeneous conformal modes are just the homothetic scales, which span a 3-dimensional minisuperspace underlying the conformally equivariant quantization scheme [38], [39], yielding the Wheeler-deWitt equation for a given point of the moduli space of local homogeneous 3-geometries. Global properties of the homogeneous 3-geometry have not considered here. However it should be clear that a given global geometry according to one of the Thurston types exists only for a specific local geometries specified by characteristic points in our moduli space (cf. [70]).

The short distance regime of quantum gravity might be described in terms of connection dynamics, recently also related to spin networks [71]. While the standard theory is worked out for the compact structure group $SU(2)$, the topology of K^3 suggests that this structure group could change by a transition to the noncompact group $E(2)$ and similarly further, until the 3-dimensional Abelian group is reached. It remains an interesting question, what happens to connections, and moreover to the holonomy groups, under such a deformation. In [71] a q-deformation of $SU(2)$ was suggested, in order to regularize infrared divergences. However, infrared divergences are obviously related to the macroscopic limit. Since in this limit the discrete structure of space-time is expected to become replaced by a continuous structure, we have no reason to expect that original spin network, or some related braid structures, might be pertained in the macroscopic theory. So a transition *within* the category of Lie algebras is more likely to provide the solution of the infrared problem, even more, since K^3-transitions appear naturally in the cosmological evolution (e.g. related to the isotropization of a homogeneous universe).

It is remarkable that, the index technique provides a kind of Morse-like potential J on K^n determining the stability and path of evolution of the isometries under consideration. For $n = 3$, the $SU(2)$ isometry is not be protected against spontaneous transitions to a "lower" isometry. In a dynamical evoltion of the homogeneous geometry it would be only *metastable*, in the sense that it would and "decay" after some time to a Bianchi isometry in K^3 which has a lower potential level, and so on, until the minimal state of Abelian symmetry is reached.

The metastability of a higher level in the potential is due to the higher dimension of the corresponding submanifold of local homogeneous moduli with just that transitive isometry group. In this this sense, isometries corresponding to interior points of that submanifold show some rigidity against transitions, due to their distance from the boundary of the submanifold.

Note that for symmetries in K^3, the dimensionality of the subspace of corresponding moduli increases with the level of the potential.

The complete parametrization of local 3-geometries of a definite class like the homogeneous one is of particular interest for a systematic approach to canonical quantization. In spatially homogeneous quantum cosmology, geometries of different isometry are usually treated separately.

6. Multidimensional structures

Let r denote the degree of differentiability of the category C^r of manifolds and C^r-homomorphisms of manifolds. For $r = 0, \omega, \infty$, and $r \geq 1$, the category C^r , its manifolds and its homomorphisms are called topological, analytical, smooth, and differentiable respectively.

A C^r manifold can carry additional structure. In particular it can admit a non-trivial multidimensional decomposition, it can carry causal structure of different strength, and it can carry geometrical structures, like a connection, and more particularly the latter may be the Levi-Civita metric one.

Any finite-dimensional C^r manifold M admits a decomposition as a direct product of C^r manifolds. The decomposition is called minimal if none of the factors itself admits a non-trivial decomposition. The unique minimal decomposition $M = M_1 \times \ldots \times M_n$ is called the multidimensional structure of M. If $n \geq 2$, M is called a multidimensional manifold.

A C^r-fiber bundle of manifolds is an exact sequence

$$M \hookrightarrow N \rightarrow \overline{M}_0, \tag{6.1}$$

with fiber space M, total space N, and base space $\overline{M}_0$ all being C^r-manifolds.

Definition: Let a C^r-fiber bundle (6.1) of manifolds be such that the fiber manifold is a direct product of length l ($1 \leq l \leq \infty$),

$$M := \times_{i=1}^{l} M_i, \tag{6.2}$$

where M_i, $i \in \mathbb{N}$, are C^r-manifolds of finite dimension d_i, and also $dim \overline{M}_0 =: D_0 < \infty$. Then the total space N of (6.1) is called a *multidimensional C^r-manifold*.

The manifolds M_i, $i \geq 1$, are called the *factor spaces* of N, and $\overline{M}_0$ denotes the base space. (Later, for considerations of dynamics and cosmology, the base manifold will be assumed to factorize, $\overline{M}_0 := \mathbb{R} \times M_0$.)

On multidimensional manifolds of finite length l there is an extensive literature, to which [23] and reference therein give somes introduction from a more contemporary point of view. Nevertheless for generality we may consider also the case of countably many factor spaces, $l = \infty$, at least in those parts which deal with the general structure of multidimensional geometry. (Examples considered currently as physically relevant for Einstein gravity, restrict to $l < \infty$ and $D_0 = 4$.)

The MD manifold N is called *internally homogeneous* if there exists a direct product group $G := \bigotimes_{i=0}^{n} G_i$ with a direct product realization $\tau := \otimes_{i=0}^{n} \tau_i$ on $\mathrm{Diff}(M) := \bigotimes_{i=0}^{n} \mathrm{Diff}(M_i)$ such that for $i = 0, \ldots, n$ the realization

$$\tau_i : G_i \rightarrow \mathrm{Diff}(M_i) \tag{6.3}$$

yields a transitive action of $\tau_i(G_i)$ on M_i.

Now for $i = 0, \ldots, n$, let each factor space M_i be equipped with a smooth homogeneous metric $g^{(i)}$. rendering it into a homogeneous Riemannian manifold. Furthermore, let $\overline{M}_0$ be equipped with an arbitrary $\mathcal{C}^{\infty}$-metric $\overline{g}^{(0)}$, and let $\overline{\gamma}$ and β^i , $i = 1, \ldots, n$ be smooth scalar fields on $\overline{M}_0$.

Then, under any projection $\mathrm{pr} : N \rightarrow \overline{M}_0$ a pullback of $e^{2\overline{\gamma}} \overline{g}^{(0)}$ from $x \in \overline{M_0}$ to $z \in \mathrm{pr}^{-1}\{x\} \subset M$, consistent with the fiber bundle (6.1) and the homogeneity of internal spaces, is given by

$$g_{(z)} := e^{2\overline{\gamma}(x)} \overline{g}^{(0)}_{(x)} \oplus_{i=1}^{n} e^{2\beta^i(x)} g^{(i)}. \tag{6.4}$$

The function $\overline{\gamma}$ fixes a *gauge* for the (Weyl) *conformal frame* on $\overline{M_0}$, corresponding just to a particular choice of geometrical variables.

$\overline{\gamma}$ uniquely defines the form of the effective D_0-dimensional theory. For example $\overline{\gamma} := 0$ defines the Brans-Dicke frame.

Let us now consider a multidimensional manifold N (6.1) of dimension $D = D_0 + \sum_{i=0}^{n} d_i$, equipped with a (pseudo) Riemannian metric (6.4) where

$$g^{(i)} \equiv g_{m_i n_i}(y_i) dy_i^{m_i} \otimes dy_i^{n_i}, \tag{6.5}$$

are R-homogeneous Riemannian metrics on M_i (i.e. the Ricci scalar $R[g^{(i)}] \equiv R_i$ is a constant on M_i), in coordinates $y_i^{n_i}$, $n_i = 1, \ldots, d_i$, and

$$x \mapsto \overline{g}^{(0)}(x) = \overline{g}^{(0)}_{\mu\nu}(x) dx^{\mu} \otimes dx^{\nu} \tag{6.6}$$

yielding a general, not necessarily R-homogeneous, (pseudo) Riemannian metric on $\overline{M}_0$.

Eq. (6.4) is the multidimensional generalization of the warped product of Riemannian manifolds from [48], namely $N = \overline{M}_0 \times_a M$, where $a := e^{\beta}$ is now a *vector-valued* root warping function, given by

$$\beta := \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_n \end{pmatrix}. \tag{6.7}$$

Below sometimes, in particular for physical application to the $\mathrm{Diff}(\overline{M}_0)$-invariant case with $D_0 = 4$, we will assume the $i = 0$ geometry to be empty and omit corresponding empty contributions to tensors, summations etc. For later convenience we also define

$$\varepsilon(I) := \prod_{i\in I} \varepsilon_i; \qquad \sigma_0 := \sum_{i=0}^{n} d_i\beta_i, \qquad \sigma_1 := \sum_{i=1}^{n} d_i\beta_i, \qquad \sigma(I) := \sum_{i\in I} d_i\beta_i, \tag{6.8}$$

where $\varepsilon_i := \mathrm{sign}(|g^{(i)}|)$ and $M_i \subset M$ for $i = 0, \ldots, n$ are all homogeneous factor spaces. Here and below, we use the shorthand $|g| := |\det(g_{MN})|$, $|\overline{g}^{(0)}| := |\det(\overline{g}^{(0)}_{\mu\nu})|$, and analogously for all other metrics including $g^{(i)}$, $i = 1, \ldots, n$.

Further, a $\overline{g}^{(0)}$-covariant derivative of a given function α w.r.t. x^μ is denoted by $\alpha_{;\mu}$, its partial derivative also by $\alpha_{,\mu}$, and $(\partial\alpha)(\partial\beta) := \overline{g}^{(0)\mu\nu}\alpha_{,\mu}\beta_{,\nu}$.

Lemma: On $\overline{M}_0$, the Laplace-Beltrami operator $\Delta[\overline{g}^{(0)}] := \frac{1}{\sqrt{|\overline{g}^{(0)}|}}\frac{\partial}{\partial x^\mu}\left(\sqrt{|\overline{g}^{(0)}|}\overline{g}^{(0)\mu\nu}\frac{\partial}{\partial x^\nu}\right)$, transforms under the conformal map $\overline{g}^{(0)} \mapsto e^{2\overline{\gamma}}\overline{g}^{(0)}$ according to

$$\begin{aligned}\Delta[e^{2\overline{\gamma}}\overline{g}^{(0)}] - e^{-2\overline{\gamma}}\Delta[\overline{g}^{(0)}] &= -e^{-2\overline{\gamma}}\overline{g}^{(0)\mu\nu}\left(\Gamma[e^{2\overline{\gamma}}\overline{g}^{(0)}] - \Gamma[\overline{g}^{(0)}]\right)^{\lambda}_{\mu\nu}\frac{\partial}{\partial x^\lambda} \\ &= e^{-2\overline{\gamma}}(D_0-2)g^{(0)\mu\nu}\frac{\partial\overline{\gamma}}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\end{aligned} \tag{6.9}$$

where Γ denotes the Levi-Civita connection.

In general, the Levi-Civita connection Γ corresponding to (6.4) does *not* decompose multidimensionally, and neither does the Riemann tensor. The latter is a section in $\mathfrak{T}^1_3 M$ which is not given as a pullback to $\overline{M}_0$ of a section in the direct sum $\oplus_{i=1}^{n}\mathfrak{T}^1_3 M_i$ of corresponding tensor bundles over the factor manifolds.

However, with (6.4) the Ricci tensor decomposes again multidimensionally.

Theorem:

$$\mathrm{Ric}[g] = \mathrm{Ric}^{(0)}[g^{(0)}, \overline{\gamma}; \phi] \oplus_{i=1}^{n} \mathrm{Ric}^{(i)}[g^{(0)}, \overline{\gamma}; g^{(i)}, \phi], \tag{6.10}$$

where

$$\begin{aligned}\mathrm{Ric}^{(0)}_{\mu\nu} &:= R_{\mu\nu}[g^{(0)}] + g^{(0)}_{\mu\nu}\Big\{-\triangle[g^{(0)}]\overline{\gamma} + (2-D_0)(\partial\overline{\gamma})^2 - \partial\overline{\gamma}\sum_{j=1}^{n} d_j\partial\phi^j\Big\} \\ &\quad +(2-D_0)(\overline{\gamma}_{;\mu\nu} - \overline{\gamma}_{,\mu}\overline{\gamma}_{,\nu}) - \sum_{i=1}^{n} d_i(\phi^i_{;\mu\nu} - \phi^i_{,\mu}\overline{\gamma}_{,\nu} - \phi^i_{,\nu}\overline{\gamma}_{,\mu} + \phi^i_{,\mu}\phi^i_{,\nu}), \\ \mathrm{Ric}^{(i)}_{m_in_i} &:= R_{m_in_i}[g^{(i)}] - e^{2\phi^i-2\overline{\gamma}}g^{(i)}_{m_in_i}\Big\{\triangle[g^{(0)}]\phi^i + (\partial\phi^i)[(D_0-2)\partial\overline{\gamma} + \sum_{j=1}^{n} d_j\partial\phi^j]\Big\}, \\ &\quad i = 1, \ldots, n,\end{aligned} \tag{6.11}$$

The corresponding Ricci curvature scalar reads

$$\begin{aligned}R[g] &= e^{-2\overline{\gamma}}R[\overline{g}^{(0)}] + \sum_{i=1}^{n} e^{-2\beta^i}R[g^{(i)}] - e^{-2\overline{\gamma}}\overline{g}^{(0)\mu\nu}\Bigg((D_0-2)(D_0-1)\frac{\partial\overline{\gamma}}{\partial x^\mu}\frac{\partial\overline{\gamma}}{\partial x^\nu} \\ &\quad + \sum_{i,j=1}^{n}(d_i\delta_{ij} + d_id_j)\frac{\partial\beta^i}{\partial x^\mu}\frac{\partial\beta^j}{\partial x^\nu} + 2(D_0-2)\sum_{i=1}^{n} d_i\frac{\partial\overline{\gamma}}{\partial x^\mu}\frac{\partial\beta^i}{\partial x^\nu}\Bigg) \\ &\quad -2e^{-2\overline{\gamma}}\Delta[\overline{g}^{(0)}]\left((D_0-1)\overline{\gamma} + \sum_{i=1}^{n} d_i\beta^i\right).\end{aligned} \tag{6.12}$$

Let us now set

$$f \equiv f[\overline{\gamma}, \beta] := (D_0 - 2)\overline{\gamma} + \sum_{j=1}^{n} d_j \beta^j, \tag{6.13}$$

where β is the vector field with the dilatonic scalar fields β^i as components. (Note that f can be resolved for $\overline{\gamma} \equiv \overline{\gamma}[f, \beta]$ if and only if $D_0 \neq 2$. The singular case $D_0 = 2$ is discussed in [26].) Then, (6.12) can also be written as

$$\begin{aligned} R[g] \quad &- \quad e^{-2\overline{\gamma}} R[\overline{g}^{(0)}] - \sum_{i=1}^{n} e^{-2\beta^i} R_i \ = \qquad\qquad (6.14)\\ = \quad &- \quad e^{-2\overline{\gamma}} \left\{ \sum_{i=1}^{n} d_i (\partial \beta^i)^2 + (\partial f)^2 + (D_0 - 2)(\partial \overline{\gamma})^2 + 2\Delta[\overline{g}^{(0)}](f + \overline{\gamma}) \right\} \\ = \quad &- \quad e^{-2\overline{\gamma}} \left\{ \sum_{i=1}^{n} d_i (\partial \beta^i)^2 + (D_0 - 2)(\partial \overline{\gamma})^2 - (\partial f)\partial(f + 2\overline{\gamma}) + R_B \right\}, \end{aligned}$$

$$R_B \ := \ \frac{1}{\sqrt{|\overline{g}^{(0)}|}} e^{-f} \partial_\mu \left[2e^f \sqrt{|\overline{g}^{(0)}|} \overline{g}^{(0)\mu\nu} \partial_\nu (f + \overline{\gamma}) \right], \tag{6.15}$$

where the last term will yield just a boundary contribution (6.22) to the action (6.21) below.

In particular it follows that the bracket $\{\cdots\}$ in (6.14) is C^r if and only if $R[g] - e^{-2\overline{\gamma}} R[\overline{g}^{(0)}] - \sum_{i=1}^{n} e^{-2\beta^i} R_i$ is C^r. The latter may be the case for Einstein manifolds with all metrics and scale factors C^r, whence the combination of derivatives in the bracket has to be C^r too.

Let us assume all M_i, $i = 1, \ldots, n$, to be connected and oriented. The Riemann-Lebesgue volume form on M_i is denoted by

$$\tau_i := \mathrm{vol}(g^{(i)}) = \sqrt{|g^{(i)}(y_i)|}\ dy_i^1 \wedge \ldots \wedge dy_i^{d_i}, \tag{6.16}$$

and the total internal space volume by

$$\mu := \prod_{i=1}^{n} \mu_i, \quad \mu_i := \int_{M_i} \tau_i = \int_{M_i} \mathrm{vol}(g^{(i)}). \tag{6.17}$$

If all of the spaces M_i, $i = 1, \ldots, n$ are compact, then the volumes μ_i and μ are finite, and so are also the numbers $\rho_i = \int_{M_i} \mathrm{vol}(g^{(i)}) R[g^{(i)}]$. However, a non-compact M_i might have infinite volume μ_i or infinite ρ_i. Nevertheless, by the R-homogeneity of $g^{(i)}$ (in particular satisfied for Einstein spaces), the ratios $\frac{\rho_i}{\mu_i} = R[g^{(i)}]$, $i = 1, \ldots, n$, are just finite constants. In any case, the D-dimensional coupling constant κ can be tuned such that, under the dimensional reduction $\mathrm{pr} : M \to \overline{M}_0$,

$$\kappa_0 := \kappa \cdot \mu^{-\frac{1}{2}} \tag{6.18}$$

becomes the D_0-dimensional physical coupling constant. If $D_0 = 4$, then ${\kappa_0}^2 = 8\pi G_N$, where G_N is the Newton constant. The limit $\kappa \to \infty$ for $\mu \to \infty$ is in particular harmless, if D-dimensional gravity is given purely by curvature geometry, without additional matter fields. If however this geometry is coupled with finite strength to additional (matter) fields, one should indeed better take care to have all internal spaces M_i, $i = 1, \ldots, n$ compact. Often this can be achieved by factorizing with an appropriate finite symmetry group.

A *conformal frame* of a smooth geometry g is a representative $\hat{g} \in [g]$ for its corresponding Weyl geometry $[g]$, i.e. there exists a smooth scalar function ω such that $g = e^{-2\omega}\hat{g}$. A parametrization of g within its Weyl class $[g]$ is given by the pair $(\hat{g}, \omega)$ of a conformal frame $\hat{g}$ and a *dilatonic* scalar ω, such that $g = e^{-2\omega}\hat{g}$. Any geometry g can be represented in different but equivalent conformal frames $(\hat{g}_1, \omega_1)$ and $(\hat{g}_2, \omega_2)$, where

$$(\hat{g}_1, \omega_1) \sim (\hat{g}_2, \omega_2) \quad :\Leftrightarrow \quad \hat{g}_1 = e^{2(\omega_1 - \omega_2)}\hat{g}_2.$$

If a geometry is taken as its own frame, its corresponding parametrization is $(\hat{g}, \omega) = (g, 0)$. Any $(\hat{g}, \omega) \neq (g, 0)$ defines a (non trivial) *reparametrization* of g within its Weyl class. Vice versa, a new conformal frame $\hat{g}$ reparametrizing g is given by a conformal transformation $g \mapsto \hat{g} := e^{2\omega}g$.

Let $\mathrm{Met}(M)$ and $\mathrm{SF}(M)$ denote the spaces of smooth metrics resp. scalar fields on M. Let $\{X(M)\}_2$ denote the 2-jet space (given by derivatives of order 0, 1 and 2) of functions in $X(M)$ w.r.t. the Levi-Civita covariant derivative operator ∇.

The numerical *value* S of the Einstein Hilbert action is invariant under reparametrization of the geometry g in a new conformal frame $\hat{g}$,

$$\begin{aligned} S &= S_{\{\mathrm{Met}(M)\}_2}[g] = S_{\{\mathrm{Met}(M)\}_2 \times \{\mathrm{SF}(M)\}_2}[\hat{g}, \omega] \ , \\ \text{where} \quad S_{\{\mathrm{Met}(M)\}_2}[g] &:= \frac{1}{2\kappa^2} \int_M d^D z \sqrt{|g|} R[g] \\ S_{\{\mathrm{Met}(M)\}_2 \times \{\mathrm{SF}(M)\}_2}[\hat{g}, \omega] &:= \frac{1}{2\kappa^2} \int_M d^D z \sqrt{|\hat{g}|} e^{(2-D)\omega} \left(R[\hat{g}] + \mathcal{L}[\hat{g}, -\omega] \right) , \end{aligned} \tag{6.19}$$

where $\mathcal{L} \equiv 0$ for $\nabla\omega = 0$. However the (Einstein Hilbert) *functional* form of the action need not be preserved even for homogeneous spaces where ω is constant. In general $S_{\{\mathrm{Met}(M)\}_2} \neq S_{\{\mathrm{Met}(M)\}_2 \times \{\mathrm{SF}(M)\}_2}$.

Nevertheless, if $S_{\{\mathrm{Met}(M)\}_2}$ is minimal at a distinguished solution geometry g_s, then, at any $\omega \in \mathrm{SF}(M)$ the restricted functional $S_{\{\mathrm{Met}(M)\}_2 \times \{\omega, \nabla\omega, \nabla^2\omega\}}$ is likewise minimal at the geometry $\hat{g}_s = e^{2\omega}g_s$, with same minimal value S. Hence, the scale field ω is a *gauge* field corresponding to the reparametrization invariance of S.

In the special case $D = 2$ (6.19) becomes purely topological modulo a boundary contribution, which vanishes for $\partial M = 0$ whence also the action functional of (6.19) is independent of ω and therefore reparametrization invariant.

On the other hand, a conformal transformation $g \mapsto \hat{g} := e^{2\omega}g$ in general also changes the value of the action functional,

$$\begin{aligned} S = \frac{1}{2\kappa^2} \int_M d^D z \sqrt{|g|} R[g] \quad \mapsto \quad \hat{S} &:= \frac{1}{2\kappa^2} \int_M d^D z \sqrt{|\hat{g}|} R[\hat{g}] \\ &= \frac{1}{2\kappa^2} \int_M d^D z \sqrt{|g|} e^{(D-2)\omega} \left(R[g] + \mathcal{L}[g, \omega] \right) . \end{aligned} \tag{6.20}$$

Similar holds for other action functionals which contain additional (boundary, scalar, and other) terms besides the Einstein Hilbert one.

6.1 Effective σ-model for pure multidimensional geometry

With the total dimension D , κ^2 a D-dimensional gravitational constant we consider a purely gravitational action of the form

$$S = \frac{1}{2\kappa^2}\int_N d^D z\sqrt{|g|}\{R[g]\} + S_{\rm GHY}. \tag{6.21}$$

Here a (generalized) Gibbons-Hawking-York [72], [73] type boundary contribution $S_{\rm GHY}$ to the action is taken to cancel boundary terms.

Lemma: In (6.21) boundary terms cancel, if and only if

$$\begin{aligned} S_{\rm GHY} &= \frac{1}{2\kappa^2}\int_N d^D z\sqrt{|g|}\{e^{-2\overline{\gamma}}R_B\} \\ &= \frac{1}{\kappa_0^2}\int_{\overline{M}_0} d^{D_0}x\frac{\partial}{\partial x^\lambda}\left(e^f\sqrt{|\overline{g}^{(0)}|}\overline{g}^{(0)\lambda\nu}\frac{\partial}{\partial x^\nu}(f+\overline{\gamma})\right). \end{aligned} \tag{6.22}$$

Proof: Eqs.(6.14) and (6.15) show that $S_{\rm GHY}$ should be taken in the form (6.22).

(6.22) is a pure boundary term in form of an effective D_0-dimensional flow through $\partial\overline{M}_0$.

After dimensional reduction the action (6.21) reads

$$\begin{aligned} S = \frac{1}{2\kappa_0^2}\int_{\overline{M}_0} d^{D_0}x\sqrt{|\overline{g}^{(0)}|}e^f\Bigg\{ & R[\overline{g}^{(0)}] + (\partial f)(\partial[f+2\overline{\gamma}]) - \sum_{i=1}^n d_i(\partial\beta^i)^2 \\ & -(D_0-2)(\partial\overline{\gamma})^2 + e^{2\overline{\gamma}}\left[\sum_{i=1}^n e^{-2\beta^i}R_i\right]\Bigg\}, \end{aligned} \tag{6.23}$$

where e^f is a dilatonic scalar field coupling to the D_0-dimensional geometry on $\overline{M}_0$.

According to the considerations above, due to the conformal reparametrization invariance of the geometry on $\overline{M}_0$, we should fix a conformal frame on $\overline{M}_0$. But then in (6.23) $\overline{\gamma}$, and with (6.13) also f, is no longer independent from the vector field β, but rather

$$\overline{\gamma} \equiv \overline{\gamma}[\beta]\ , \qquad f \equiv f[\beta]. \tag{6.24}$$

Then, modulo the conformal factor e^f, the dilatonic kinetic term of (6.23) takes the form

$$(\partial f)(\partial[f+2\overline{\gamma}]) - \sum_{i=1}^n d_i(\partial\beta^i)^2 - (D_0-2)(\partial\overline{\gamma})^2 = -G_{ij}(\partial\beta^i)(\partial\beta^j), \tag{6.25}$$

with $G_{ij} \equiv {}^{(\overline{\gamma})}G_{ij}$, where

$$ {}^{(\overline{\gamma})}G_{ij} := {}^{(\rm BD)}G_{ij} - (D_0-2)(D_0-1)\frac{\partial\overline{\gamma}}{\partial\beta^i}\frac{\partial\overline{\gamma}}{\partial\beta^j} - 2(D_0-1)d_{(i}\frac{\partial\overline{\gamma}}{\partial\beta^{j)}}, \tag{6.26}$$

$$ {}^{(\rm BD)}G_{ij} := \delta_{ij}d_i - d_i d_j. \tag{6.27}$$

(In (6.26) the brackets $(\cdots)$ denote symmetrization.) For $D_0 \neq 2$, we can write equivalently $G_{ij} \equiv {}^{(f)}G_{ij}$, where

$$ {}^{(f)}G_{ij} := {}^{(\rm E)}G_{ij} - \frac{D_0-1}{D_0-2}\frac{\partial f}{\partial\beta^i}\frac{\partial f}{\partial\beta^j}, \tag{6.28}$$

$$ {}^{(\rm E)}G_{ij} := \delta_{ij}d_i + \frac{d_i d_j}{D_0-2}. \tag{6.29}$$

For $D_0 = 1$, $G_{ij} = {}^{(E)}G_{ij} = {}^{(BD)}G_{ij}$ is independent of $\overline{\gamma}$ and f. Note that the metrics (6.27) and (6.29) (with $D_0 \neq 2$) may be diagonalized to $(\mp(\pm)^{\delta_{1D_0}})^{\delta_{1i}}\delta_{ij}$ respectively, by homogeneous linear minisuperspace coordinate transformations $\beta \overset{T}{\mapsto} z$ and $\beta \overset{Q}{\mapsto} \varphi$, explicitly given by components

$$z^1 := {}^{(BD)}q^{-1} \sum_{j=1}^{n} d_j \beta^j \ , \qquad \varphi^1 := {}^{(E)}q^{-1} \sum_{j=1}^{n} d_j \beta^j \ ,$$

$$z^i \equiv \varphi^i \quad := \quad \left[d_{i-1}/\, \Sigma_{i-1}\Sigma_i\right]^{1/2} \sum_{j=i}^{n} d_j \left(\beta^j - \beta^{i-1}\right) \ , \tag{6.30}$$

$i = 2, \ldots, n$, where with $D' := D - D_0$ and $\Sigma_k := \sum_{i=k}^{n} d_i$,

$$^{(BD)}q := \sqrt{\frac{D'}{D'-1}} \ , \qquad ^{(E)}q := \sqrt{\frac{D'(D_0-2)}{D'+D_0-2}} \ . \tag{6.31}$$

So, after fixing a conformal reparametrization gauge for the geometry on M_0, (6.21) becomes a σ-model, where the vector field β (or z resp. φ) defines the coordinates of its n-dimensional target space. In the following, we will simplify notation by a summation convention for tensors over target space.

In general, for $n > 2$ and non-constant functional $\gamma[\beta]$, the minisuperspace metric given by (6.25) and the conformally related target space metric may not even be conformally flat. However, for constant $\overline{\gamma}$, (6.26) reduces to (6.27), whence target space is conformally flat, namely it is related to n-dimensional Minkowski space by a conformal scale factor

$$\varphi \equiv \varphi(\beta) := \prod_{l=1}^{n} e^{d_l \beta^l} = e^{{}^{(BD)}q z^1} = e^{{}^{(E)}q \varphi^1}, \tag{6.32}$$

which is proportional to the total internal space volume.

In the case $D_0 \neq 2$, for non-constant functional $f[\beta]$, the target space may again in general not be conformally flat for $n > 2$. However, for constant f, (6.28) reduces to (6.29), whence, target space is a flat n-dimensional space, namely an Euclidean one for $D_0 > 2$, and a Minkowskian one for $D_0 = 1$.

After gauging $\overline{\gamma}$, setting $m := \kappa_0^{-2}$, (6.23) yields a σ-model in the form

$$^{(\overline{\gamma})}S = \int_{\overline{M}_0} d^{D_0}x \sqrt{|\overline{g}^{(0)}|}\, {}^{(\overline{\gamma})}N^{D_0} \quad \varphi(\beta) \ \Big\{ \frac{m}{2} {}^{(\overline{\gamma})}N^{-2} \left[R[\overline{g}^{(0)}] - {}^{(\overline{\gamma})}G_{ij}(\partial\beta^i)(\partial\beta^j)\right] - {}^{(BD)}V(\beta) \Big\} \ , \tag{6.33}$$

$$\text{where} \qquad {}^{(BD)}V(\beta) \ := \ m \left[-\frac{1}{2} \sum_{i=1}^{n} R[g^{(i)}] e^{-2\beta^i} \right] \ , \tag{6.34}$$

$$^{(\overline{\gamma})}N \ := \ e^{\overline{\gamma}} \ . \tag{6.35}$$

Note that, the potential (6.34) and the conformal factor $\phi(\beta) := \prod_{i=1}^{n} e^{d_i \beta^i}$ are gauge invariant.

Analogously, from (6.23) a σ-model action can be obtained for each gauge f.

Lemma:

$$
{}^{(f)}S = \int_{\overline{M}_0} d^{D_0}x\sqrt{|\overline{g}^{(0)}|}\,{}^{(f)}N^{D_0} \quad \Big\{\frac{m}{2}{}^{(f)}N^{-2}\left[R[\overline{g}^{(0)}] - {}^{(f)}G_{ij}(\partial\beta^i)(\partial\beta^j)\right] - {}^{(\mathrm{E})}V(\beta)\Big\}\ , \tag{6.36}
$$

$$
{}^{(\mathrm{E})}V(\beta) \ := \ m\Omega^2\left[-\frac{1}{2}\sum_{i=1}^{n} R[g^{(i)}]e^{-2\beta^i}\right]\ , \tag{6.37}
$$

$$
{}^{(f)}N \ := \ e^{\frac{f}{D_0-2}}\ , \tag{6.38}
$$

where the function Ω on $\overline{M}_0$ is defined as

$$
\Omega := \varphi^{\frac{1}{2-D_0}}\ . \tag{6.39}
$$

Note that, with Ω also the potential (6.37) is gauge invariant, and the dilatonic target-space, though not even conformally flat in general, is flat for constant f.

In fact, Eqs. (6.33)-(6.35) and (6.36)-(6.38) show that there are at least two special frames.

The first one corresponds to the gauge $\overline{\gamma} \stackrel{!}{=} 0$. In this case ${}^{(\overline{\gamma})}N = 1$, the minisuperspace metric (6.26) reduces to the Minkowskian (6.27), the dilatonic scalar field becomes proportional to the internal space volume, $e^{f[\beta]} = \varphi(\beta) = \prod_{i=1}^n e^{d_i\beta^i}$, and (6.33) describes a generalized σ-model with conformally Minkowskian target space. The Minkowskian signature implies a negative sign in the dilatonic kinetic term. This frame is usually called the Brans-Dicke one, because $\varphi = e^f$ here plays the role of a Brans-Dicke scalar field.

The second distinguished frame corresponds to the gauge $f \stackrel{!}{=} 0$, where $\overline{\gamma} = \frac{1}{2-D_0}\sum_{i=1}^n d_i\beta^i$ is well-defined only for $D_0 \neq 2$. In this case ${}^{(f)}N = 1$, the minisuperspace metric (6.28) reduces to the Euclidean (6.29), and (6.36) describes a self-gravitating σ-model with Euclidean target space. Hence all dilatonic kinetic terms have positive signs. This frame is usually called the Einstein one, because it describes an effective D_0-dimensional Einstein theory with additional minimally coupled scalar fields. For multidimensional geometries with $D_0 = 2$ the Einstein frame fails to exist, which reflects the well-known fact that two-dimensional Einstein equations are trivially satisfied without implying any dynamics.

For $D_0 = 1$, the action of both (6.33) and (6.36) was shown in [32] (and previously in [38], [39]) to take the form of a classical particle motion on minisuperspace, whence different frames correspond are just related by a time reparametrization. More generally, for $D_0 \neq 2$ and $(\overline{M}_0, \overline{g}(0))$ a vacuum space-time, the σ-model (6.36) with the gauge $f \stackrel{!}{=} 0$ describes the dynamics of a massive (D_0-1)-brane within a potential (6.37) on its target minisuperspace.

In fact, the target space is in general a conformally homogeneous space, and in the Einstein frame a homogeneous one. Once its isometry group $\mathfrak{G}$ and isotropy group $\mathfrak{H}$ are known, it is clear that the sigma model (6.36) can also be written in matrix form

Lemma:

$$
{}^{(f)}S = \int_{\overline{M}_0} d^{D_0}x\sqrt{|\overline{g}^{(0)}|}N^{D_0}(\mathcal{M}) \quad \Big\{\frac{m}{2}N^{-2}(\mathcal{M})\left[R[\overline{g}^{(0)}] + g^{(0)\mu\nu}B\mathrm{Tr}_{\,\rho}(\partial_\mu\mathcal{M}\partial_\nu\mathcal{M}^{-1})\right] - {}^{(\mathrm{E})}U(\mathcal{M})\Big\}\ , \tag{6.40}
$$

with $\mathcal{M} \in \rho(G)$ where ρ is an appropriate coset representation of the target space $\mathfrak{M} := \mathfrak{G}/\mathfrak{H}$, ${}^{(\mathrm{E})}U$ is now the corresponding potential on $\mathfrak{M}$, N a gauge function on $\mathfrak{M}$, and B a normalization.

For $D_0 = 4$, eq. (6.40) can also be written in the Einstein frame as

$$ {}^{(E)}S = \int_{\overline{M}_0} \Big\{ \frac{m}{2} \left[\mathrm{Tr}\, \Omega \wedge *\Sigma + B \mathrm{Tr}\,_\rho d\mathcal{M} \wedge *d\mathcal{M}^{-1} \right] - {}^{(\mathrm{E})}U(\mathcal{M}) * 1 \Big\} \ , \tag{6.41} $$

where Ω is the curvature 2-form, $\Sigma := e \wedge e$ and $\overline{g}^{(0)}$ are given by the D_0-dimensional soldering 1-form e, and the Hodge star is taken w.r.t. $(\overline{M}, g^{(0)})$. The form (6.41) is a then a convenient starting point for the canonical quantization procedure.

In the purely gravitational model consider so far $\mathfrak{M}$ is a finite dimensional and homogeneous with a transitive Abelian group. In the following section let us add minimally coupled scalar and $p+2$-form matter fields and investigate the extension of the resulting target space $\mathfrak{M}$.

Note: In the following sections the base space $\overline{M}_0$ is denoted for simplicity just as M_0, unless for those solutions in section 7 below where $\overline{M}_0$ or its geometry decomposes.

6.2 σ-model with extra scalars and $p+2$-forms

We now couple the purely gravitational action (6.21) to additional matter fields of scalar and generalized Maxwell type, i.e. we consider now the action

$$ 2\kappa^2 [S[g,\phi,F^a] - S_{\mathrm{GHY}}] = \int_N d^D z \sqrt{|g|} \{ R[g] - C_{\alpha\beta} g^{MN} \partial_M \phi^\alpha \partial_N \phi^\beta - \sum_{a\in\Delta} \frac{\eta_a}{n_a!} \exp[2\lambda_a(\phi)](F^a)^2 \} \tag{6.42} $$

of a self-gravitating σ model on M with topological term S_{GHY}. Here the l-dimensional target space, defined by a vector field ϕ with scalar components ϕ^α, $\alpha = 1, \ldots, l$, is coupled to several antisymmetric n_a-form fields F^a via 1-forms λ_a, $a \in \Delta$. For consistency, we have to demand of course that all fields are internally homogeneous. We will see below how this gives rise to an effective $l + |\Delta|$-dimensional target-space extension. Note also that for convenience here we work with fields ϕ and F which differ from the actual (physical) matter fields by a rescaling with the square root of the coupling constant.

With $I \subset \{1, \ldots, n\}$, the generalized Maxwell fields F^a are located on $(n_a - 1)$-dimensional world sheets

$$ M_I := \prod_{i\in I} M_i = M_{i_1} \times \ldots \times M_{i_k}, \tag{6.43} $$

$$ n_a - 1 = D(I) := \sum_{i\in I} d_i = d_{i_1} + \ldots + d_{i_k}. \tag{6.44} $$

of different $(n_a - 2)$-branes, labeled for each a by the sets I in a certain subset Ω_a of the power set of $\{1, \ldots, n\}$. Variation of (6.42) yields the field equations

$$\begin{aligned} R_{MN} - \frac{1}{2} g_{MN} R &= T_{MN}, & (6.45)\\ C_{\alpha\beta}\Delta[g]\phi^\beta - \sum_{a\in\Delta} \frac{\eta_a \lambda_a^\alpha}{n_a!} e^{2\lambda_a(\phi)}(F^a)^2 &= 0, & (6.46)\\ \nabla_{M_1}[g](e^{2\lambda_a(\phi)} F^{a,M_1 M_2 \ldots M_{n_a}}) &= 0, & (6.47)\end{aligned}$$

$a \in \Delta$, $\alpha = 1, \ldots, l$.

In (6.45) the D-dimensional energy-momentum resulting from (6.42) is given by a sum

$$T_{MN} := \sum_{\alpha=1}^{l} T_{MN}[\phi^\alpha, g] + \eta_a \sum_{a\in\Delta} e^{2\lambda_a(\phi)} T_{MN}[F^a, g], \tag{6.48}$$

of contributions from scalar and generalized Maxwell fields,

$$\begin{aligned} T_{MN}[\phi^\alpha, g] &:= C_{\alpha\beta}\partial_M \phi^\alpha \partial_N \phi^\alpha - \frac{1}{2} g_{MN} \partial_P \phi^\alpha \partial^P \phi^\alpha, & (6.49)\\ T_{MN}[F^a, g] &:= \frac{1}{n_a!}\left[-\frac{1}{2} g_{MN}(F^a)^2 + n_a F^a_{MM_2\ldots M_{n_a}} F^{a\ \ M_2\ldots M_{n_a}}_{\ N}\right]. & (6.50)\end{aligned}$$

We give now a sufficient criterion for the energy-momentum tensor (6.48) to decompose multidimensionally.

Let $W_1 := \{i \mid i > 0,\ d_i = 1\}$ be the label set of 1-dimensional factor spaces of the multidimensional decomposition, and set $n_1 := |W_1|$. Define

$$W(a; i, j) := \{(I, J) \mid I, J \in \Omega_a,\ (I \cap J) \cup \{i\} = I \not\ni j,\ (I \cap J) \cup \{j\} = J \not\ni i\} \tag{6.51}$$

Then the following holds.

Theorem: If for $n_1 > 1$ the p-branes satisfy the condition for all $a \in \Delta$, $i, j \in W_1$ with $i \neq j$, the condition

$$W(a; i, j) \stackrel{!}{=} \emptyset \quad \forall a \in \Delta \forall i, j \in W_1, \tag{6.52}$$

then the energy-momentum (6.48) decomposes multidimensionally without further constraints.

Proof: The only possible obstruction to the multidimensional decomposition of (6.48) comes from the second term of (6.50), $F^a_{MM_2\ldots M_{n_a}} F^{a\ \ M_2\ldots M_{n_a}}_{\ N}$ when the indices M and N take values in different index sets labeling different 1-dimensional factor spaces. The theorem then follows just from the antisymmetry of the F-fields. □

Corollary: A sufficient condition for the multidimensional decomposition of (6.48) is

$$n_1 \stackrel{!}{\leq} 1. \tag{6.53}$$

If condition (6.52) does not hold, multidimensional decomposability of (6.48) may impose additional non-trivial constraints on the $p + 2$-form fields.

Let us now specify the components of the F-fields of generalized electric and magnetic type.

Antisymmetric fields of generalized electric type, are given by scalar potential fields $\Phi^{a,I}$, $a \in \Delta$, $I \in \Omega_a$, which compose to a $(\sum_{a\in\Delta} |\Omega_a|)$-dimensional vector field Φ. Magnetic type fields are just given as the duals of appropriate electric ones.

$$F^{e,I} = d\Phi^{e,I} \wedge \tau(I) \tag{6.54}$$

$$F^{m,I} = e^{-2\lambda_a(\phi)} * (d\Phi^{m,I} \wedge \tau(J)). \tag{6.55}$$

In the Einstein frame, the action then reduces to

$$\begin{aligned} {}^{(\mathrm{E})}S[g^{(0)}, \beta, \phi, \Phi] = \int_{M_0} d^{D_0}x\sqrt{|g^{(0)}|}\Big\{\frac{m}{2}\Big[R[g^{(0)}] - G_{ij}(\partial\beta^i)(\partial\beta^j) \\ -C_{\alpha\beta}(\partial\phi^\alpha)(\partial\phi^\beta) - \sum_{a\in\Delta, I\in\Omega_a} \varepsilon_{a,I} e^{2(\lambda_a(\phi) - d_i\beta^i)}(\partial\Phi^{a,I})^2\Big] - {}^{(\mathrm{E})}V(\beta)\Big\}\ , \end{aligned} \tag{6.56}$$

which corresponds to an purely Einsteinian σ-model on M_0 with extended $(n + l + \sum_{a\in\Delta} |\Omega_a|)$-dimensional target space and dilatonic potential (6.37). Here and below we will consider by default the Einstein frame, and set correspondingly $G_{ij} := {}^{(\mathrm{E})}G_{ij}$. In (6.56) and below a summation convention is assumed also on the extended target space.

For convenience, let us introduce the topological numbers

$$l_{jI} := -\sum_{i\in I} D_i\delta^i_j, \qquad j = 1, \ldots, n, \tag{6.57}$$

and with $N := n + l$ define and define a $N \times |S|$-matrix

$$L = (L_{As}) = \begin{pmatrix} L_{is} \\ L_{\alpha s} \end{pmatrix} := \begin{pmatrix} l_{iI} \\ \lambda_{\alpha a} \end{pmatrix}, \tag{6.58}$$

a N-dimensional vector field $(\sigma^A) := (\beta^i, \phi^\alpha)$, $A = 1, \ldots, n, n + 1, \ldots, N$, composed by dilatonic and matter scalar fields, and a non-degenerate (block-diagonal) $N \times N$-matrix

$$\hat{G} = \left(\hat{G}_{AB}\right) = \begin{pmatrix} G_{ij} & 0 \\ 0 & C_{\alpha\beta} \end{pmatrix}. \tag{6.59}$$

With these definitions, (6.56) takes the form

$$S_0 = \int_{M_0} d^{D_0}x\sqrt{|g^{(0)}|}\Big\{\frac{m}{2}\Big[R[g^{(0)}] - \hat{G}_{AB}\partial\sigma^A\partial\sigma^B - \sum_{s\in S}\varepsilon_s e^{2L_{As}\sigma^A}(\partial\Phi^s)^2\Big] - {}^{(\mathrm{E})}V(\sigma)\Big\} \tag{6.60}$$

6.3 Target space structure

Theorem: The target space $(\mathfrak{M}, \mathfrak{g})$ is a homogeneous space.

Proof: The Killing vectors of a transitive subgroup of Isom$(\mathfrak{M})$ can be determined explicitly.

$$V_s := \frac{\partial}{\partial\Phi^s}, \qquad s \in S,$$

$$U_A := \frac{\partial}{\partial x^A} - \sum_{s\in S} L^s_A \Phi^s \frac{\partial}{\partial\Phi^s}, \qquad A = 1, \ldots, N. \tag{6.61}$$

Moreover, the Lie-algebra of the transitive group of isometries generated by (6.61) reads

$$\begin{aligned} [U,U] = [V,V] &= 0 \\ [U_A, V_s] &= L^s_A V_s, \qquad A = 1,\dots,N, \quad s \in S \ . \end{aligned} \tag{6.62}$$

Theorem: The target space $(\mathfrak{M}, \mathfrak{g})$ is locally symmetric if and only if $< L^s, L^r >_{\hat{G}} (L^s - L^r) = 0 \ \forall r, s \in S$.

Proof: Let $\mathfrak{R}iem$ denote the Riemann tensor of $(\mathfrak{M}, \mathfrak{g})$. The latter is locally symmetric, if and only if

$$\nabla \mathfrak{R}iem = 0, \tag{6.63}$$

where ∇ denotes the covariant derivative w.r.t. $\mathfrak{g}$. However, the only non-trivial equations (6.63) are

$$\nabla_p \mathfrak{R}_{srqA} = k_{psrq} < L^s, L^r >_{\hat{G}} (L^r_A - L^s_A) = 0, \quad A = 1,\dots,N, \quad p,q,r,s \in S \tag{6.64}$$

with $k_{psrq} := \varepsilon_s \varepsilon_r e^{2L^s + 2L^r} (\delta_{ps}\delta_{rq} + \delta_{pr}\delta_{sq})$ nonzero for fixed s, r.

6.4 Special coordinate gauges on M_0

In this section the existence and use of some special coordinates on M_0 is investigated. If these coordinates exist, they give, in particular, a specific meaning to the particular reparametrization gauges of constant γ resp. constant f above. Let us consider a metric on M_0 denoted as

$$g = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu, \tag{6.65}$$

with coordinates x^ν, $\nu = 0, \dots, D_0 - 1$. A special standard representation of g is given, if there exist coordinates t^μ, $\mu = 0, \dots, D_0 - 1$ such that

$$g = \eta_{\mu\nu} dt^\mu \otimes dt^\nu, \tag{6.66}$$

with $\eta_{\mu\nu} = \eta_\mu \delta_{\mu\nu}$, $\eta_\mu \in \{-1, +1\}$.

The standard representation (6.66) exists if and only if g is flat with arbitrary signature (η_μ). In this case coordinates t^μ are called *proper* coordinates w.r.t. g on M_0.

Let us now find the proper coordinates t^μ_X for a flat metric of arbitrary signature in arbitrary a priori coordinates x^μ,

$$g_X = g_{X\,\mu\nu}(x) dx^\mu \otimes dx^\nu \stackrel{!}{=} \eta_{\mu\nu} dt^\mu_X \otimes dt^\nu_X, \tag{6.67}$$

We decompose the coordinate transformation $x^\mu \mapsto t^\mu_X$ in two steps. First, we diagonalize $g_{X\,\mu\nu}(x)$ by a local SO$(N, D_0 - N)$ rotation $L_X(x)$ in the cotangent space $T^*_x M_0$ of each point $x \in M_0$, $dx^\mu \mapsto dy^\mu := L^\mu_{X\,\nu}(x) dx^\nu$, whence

$$g_X = \frac{g_{X\,\mu\mu}(y)}{\eta_\mu} \eta_{\mu\nu} dy^\mu \otimes dy^\nu \stackrel{!}{=} \eta_{\mu\nu} dt^\mu_X \otimes dt^\nu_X. \tag{6.68}$$

Then, with $S_X(y) \in \mathrm{Diag}_+(D_0)$ defined by $S_X{}^\mu_\nu(y) := \sqrt{\frac{g_{X\mu\mu}(y)}{\eta_{\mu\mu}}}\delta^\mu_\nu$, we perform a local rescaling $dy^\mu \mapsto dt^\mu_X := S_X{}^\mu_\nu(y)dy^\nu = \sqrt{\frac{g_{X\mu\mu}(y)}{\eta_\mu}}dy^\mu$. This yields the proper coordinate vector $t_X = \int S_X(y(x)) \cdot L_X(x) \cdot dx$ with explicit components

$$t^\mu_X = \int \sqrt{\frac{||((L_X(x))^{-1})_\mu||^2_X}{\eta_\mu}} L^\mu_{X\,\nu} dx^\nu. \tag{6.69}$$

Here and in the following, for a given metric (frame) g_X, $||A_\mu||^2_X :=< A_\mu, A_\mu >_X$, where $< A_\mu, A_\nu >_X := g_{X\alpha\beta}A^\alpha{}_\mu A^\beta{}_\nu$, and A_μ is the vector of column μ in A.

Now any other metric $g_Y = g_{Y\mu\nu}(x)dx^\mu \otimes dx^\nu$, in the same a priori coordinates x^μ, can be expressed in g_X proper coordinates as $g_Y = g_{Y\mu\nu}(t_X)dt^\mu_X \otimes dt^\nu_X$ with components

$$g_{Y\mu\nu}(t_X) = g_{Y\alpha\beta}(x)G^{\alpha\beta}_{X\,\mu\nu}(x), \tag{6.70}$$

$$G^{\alpha\beta}_{X\,\mu\nu}(x) := \sqrt{\frac{||(L_X(x))_\alpha||^2_X ||(L_X(x))_\beta||^2_X}{\eta_{\alpha\alpha}\eta_{\beta\beta}}}((L_X(x))^{-1})^\alpha{}_\mu((L_X(x))^{-1})^\beta{}_\nu,$$

where $x \equiv x(t_X)$ is given by inversion of (6.69).

Specially interesting is the case of a pair of conformally related metrics g_X and $g_Y = e^{2\omega}g_X$. Then in general, for non-constant ω, only one of g_X and g_Y can be flat, whence proper coordinates exist only in this one.

The exceptional case is of course $D_0 = 1$, where a proper coordinate exist for any g_X, with $g_X = g_{X00}(x)dx \otimes dx \stackrel{!}{=} \eta_0 dt_X \otimes dt_X$, and the expression from (6.69) for the proper coordinate takes the familiar form

$$t_X = \int \sqrt{|g_{X00}(x)|}dx. \tag{6.71}$$

Here, if and only if g_X and g_Y have the same sign η_0, they are conformally related, i.e. $g_Y = e^{2\omega}g_X$, whence the (conformal) factor

$$e^\omega = \frac{dt_Y}{dt_X} \tag{6.72}$$

relates the respective proper coordinates. A particularly important case is t being a proper *time* coordinate, which then is also called *synchronous* time.

In particular, for a D-dimensional geometry $g = e^{2\gamma(x)}g^{(0)} + \sum_{i=1}^n e^{2\beta^i(x)}g^{(i)}$ with flat $g^{(0)}$, the homogeneous gauge $\gamma \stackrel{!}{=} 0$ of $g^{(0)}$ is called *proper (coordinate) gauge*, because with $g^{(0)} = \eta_{\mu\nu}dt^\mu \otimes dt^\nu$ the coordinates t^μ are proper for $g^{(0)}$. If t^0 is a proper time coordinate, the gauge $\gamma \stackrel{!}{=} 0$ is also called *synchronous (time) gauge* (see also [38]).

In the following, a metric on M_0 is denoted as

$$g^{(0)} = g^{(0)}{}_{\mu\nu}(x)dx^\mu \otimes dx^\nu. \tag{6.73}$$

Particular coordinates x^ν, $\nu = 0, \ldots, D_0 - 1$, are *harmonic* w.r.t. a multidimensional metric $g = e^{2\gamma(x)}g^{(0)} + \sum_{i=1}^n e^{2\beta^i(x)}g^{(i)}$, if and only if

$$0 \stackrel{!}{=} \Delta[g]x^\nu \tag{6.74}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}\left(e^{(D_0-2)\gamma+d_i\beta^i}\sqrt{|g^{(1)}|}\cdots\sqrt{|g^{(n)}|}\sqrt{|g^{(0)}|}g^{(0)\mu\nu}\right)\\
&= \frac{1}{e^{d_i\beta^i+D_0\gamma}\sqrt{|g^{(0)}|}}\frac{\partial}{\partial x^\mu}\left(e^{(D_0-2)\gamma+D_i\beta^i}\sqrt{|g^{(0)}|}g^{(0)\mu\nu}\right)\ , \quad \nu=0,\ldots,D_0-1,
\end{aligned}
$$

where the last step exploits the homogeneity of internal geometries $g^{(i)}$, $i=1,\ldots,n$.

With $f=(D_0-2)\gamma+d_i\beta^i$, the condition (6.74) reads

$$
0 \stackrel{!}{=} e^{2\gamma}\Delta[g]x^\nu = \frac{g^{(0)\nu\mu}}{\sqrt{|g^{(0)}|}}\frac{\partial f}{\partial x^\mu} - g^{(0)\mu\lambda}\Gamma[g^{(0)}]^\nu_{\mu\lambda}\ , \quad \nu=0,\ldots,D_0-1. \tag{6.75}
$$

Hence, for flat $g^{(0)}$, coordinates x^ν on M_0 are harmonic w.r.t. g, if and only if f is constant. Such a gauge, and in particular the homogeneous linear gauge $f\stackrel{!}{=}0$, is called *harmonic (coordinate) gauge*.

Under the gauge $f\stackrel{!}{=}0$, the coordinates x^ν are harmonic w.r.t. g, if and only if

$$
\frac{1}{\sqrt{|g^{(0)}|}}\frac{\partial}{\partial x^\mu}\left(\sqrt{|g^{(0)}|}g^{(0)\mu\nu}\right) = -g^{(0)\mu\lambda}\Gamma[g^{(0)}]^\nu_{\mu\lambda} \stackrel{!}{=} 0. \tag{6.76}
$$

Let us consider the case where $g^{(0)}$ is conformally flat with ${g^{(0)}}_{\mu\nu}=e^{2\omega}\eta_{\mu\nu}$. Then, condition (6.76) is satisfied if and only if

$$
\frac{\partial\omega}{\partial x^\mu} \stackrel{!}{=} 0\ , \quad \nu=0,\ldots,D_0-1, \tag{6.77}
$$

whence ω constant and $g^{(0)}$ is flat. Let coordinates on M_0 which are harmonic w.r.t. a multidimensional metric on M in the gauge $f\stackrel{!}{=}0$ be denoted by τ^ν, $\nu=0,\ldots,D_0-1$. If τ^0 is a time coordinate, the gauge $f\stackrel{!}{=}0$ is also called *harmonic time gauge*.

7. Cosmological application III: multidimensional solutions

Historically σ-models have turned out to be a very powerful tool in many areas of physics. In gravity the importance was soon realized [74] in the context of solution generating techniques [75]. More recently, σ-models have been also discussed in the context of string theory [76, 77, 78].

The purpose of this paper is to clarify the geometric structure of the effective σ-model for multidimensional Einstein geometry and to demonstrate its applicability in such different directions as cosmology, (extended) string theory, and quantization of certain higher-dimensional geometric actions.

In fact it turns out to be a very powerful tool which, on one side, allows to test the geometric content of string and M-theory down to their concrete physical imprints in the physical space-time and, on the other side, prepares a well defined class of classical higher-dimensional geometries for the canonical quantization program in dimension $D_0\le 4$ whenever this is applicable to pure Einstein gravity itself. In principle all cases with infinite number of degrees of freedom in dimension $D_0 = 4$ which can be canonically quantized have some analogous cases where additional extra dimensions add only a finite number of degrees of freedom without disturbing the integrability of the problem. These cases include of course also recently investigated midisuperspace 4-geometries. In the case of spherical symmetries, and more particular in the static case, one can find particular solutions to a classical system of the multidimensional Einstein action with scalar and antisymmetric $p+2$-form fields which are multidimensional extensions of black hole solutions. It turns out that the standard surface gravity and the Hawking temperature T_H as calculated from a Komar-like integral depend sensitively on the intersection dimension of the p-branes involved in the solution. This provides, at least in principle, an observational window to very direct geometrical properties of possible extra dimensions. Apart from that, the multidimensional σ-model contains all kinds of multidimensional spatially homogeneous cosmological models as degenerate minisuperspace cases with a finite number of degrees of freedom only.

Below, the effective D_0-dimensional σ-model is derived from a multidimensional action of Einstein type in a higher dimension D, first for pure geometry, then with additional scalar and antisymmetric $p+2$-form matter fields. The domains of the $p+1$-form potentials of the antisymmetric $p+2$-forms are the world-sheets of p-branes. In extended string and M-theory [79, 80, 81] strings are generalized to membranes as higher-dimensional objects. Most of these unified models are modeled initially on a higher-dimensional space-time manifold, say of dimension $D > 4$, which then undergoes some scheme of spontaneous compactification.

The geometric structure of the target-space is clarified. In particular it is shown that it is a always a homogeneous space. It is furthermore locally symmetric if and only if the characteristic target-space vectors satisfy a particular orthogonality condition, called the *orthobrane* relation whenever they are not identical. In any case, it turns out possible to express the *general* exact solutions in terms of elementary functions, provided the input parameters of the model satisfy the , whence the target space is locally symmetric.

Solutions of the corresponding field equations are discussed generally and with concrete examples. Particular solutions for the subcases with Ricci flat internal spaces with scalar fields only, and with intersecting p-branes are presented. In the subcase of spherically symmetric solutions the relation to particles and black p-branes is given. Although a priori one might admit all possible types of components of F-fields compatible with spherical symmetry, namely, electric, magnetic and quasiscalar ones, we concentrate on true electric and magnetic type fields, since these are the ones which admit black hole solutions.

Besides the orthobrane solutions which by now became popular in string theory, there are further families of solutions, which have another additional symmetry, e.g. coinciding F-field charges for the electro-magnetic solutions. In target space this additional symmetry is expressed by a linear relation between certain column vectors of the coupling matrix. In this case the original orthobrane conditions reduce to some weaker set of orthogonality conditions.

In the case of static, spherical symmetric solutions it is demonstrated that the formal Hawking temperature T_H (as it might appear to an observer at infinity) depends sensitively on the intersection dimension of the p-branes. Hence solutions to the multidimensional σ-model allow to detect possible imprints from extra-dimensional internal factor spaces within the physical dimension $D_0 = 4$. The black hole solutions depend on 3 integration constants, related to the electric, the magnetic, and the mass charge. It is also shown that the Hawking temperature of such black holes depends on the intersection dimension $d_{\rm int}$ of the corresponding p-branes. In an extremal limit of the charges, the black hole temperature turns out to converge to zero for $d_{\rm int} = 0$, to a finite limit for $d_{\rm int} = 1$, and to infinity for $d_{\rm int} > 1$.

Finally it is shown how the geometries of well known solutions in a Brans-Dicke frame can be transformed to the physically relevant Einstein frame.

7.1 Solution with Abelian target-space

In the this section we consider the σ-model (6.56) without the Φ fields from the $p+2$-forms, whence the target-space is the $n+l$-dimensional Abelian one, and present a particularly interesting vacuum solution.

We derive for $D_0 \neq 2$ an new exact Ricci flat multidimensional solution for the effective σ-model (6.56) in the harmonic gauge $(2 - D_0)\overline{\gamma} \stackrel{!}{=} d_i \beta^i$ with zero potential (6.37) and zero Φ. The field equation then read

$$G_{ij}\partial_\mu \beta^i \partial_\nu \beta^j + C_{\alpha\beta}\partial_\mu \phi^\alpha \partial_\nu \phi^\beta = 0, \qquad \mu,\nu = 0,\ldots,D_0-1, \tag{7.1}$$

$${}^{(\rm E)}G_{ij}\Delta[\overline{g}^{(0)}]\beta^j = 0, \qquad i = 1,\ldots,n, \tag{7.2}$$

$$C_{\alpha\beta}\Delta[\overline{g}^{(0)}]\phi^\beta = 0, \qquad \alpha = 1,\ldots,l. \tag{7.3}$$

In particular, we now solve these equations with flat $(\overline{M}_0, \overline{g}^{(0)})$. In this case, there exist g-harmonic $\overline{M}_0$-coordinates τ^μ, $\mu = 0,\ldots,D_0-1$. Let $g^{(0)} = e^{-2\overline{\gamma}}\eta_{\mu\nu}d\tau^\mu d\tau^\nu$. In such harmonic coordinates, equations (7.2) and (7.3) are solved by

$$\beta^i = b^i_\mu \tau^\mu + c^i, \qquad i = 1,\ldots,n, \tag{7.4}$$

$$\phi^\alpha = b^{n+\alpha}_\mu \tau^\mu + c^{n+\alpha}, \qquad \alpha = 1,\ldots,l. \tag{7.5}$$

We set

$$\varphi^i_\mu := \frac{\partial}{\partial \tau^\mu}\varphi^i \ , \quad \mu = 0,\ldots,D_0-1 \ . \tag{7.6}$$

With (7.4), the harmonic gauge condition reads

$$A_\mu := {}^{(\rm E)}q\varphi^1_\mu = \sum_i d_i b^i_\mu \stackrel{!}{=} 0 \ , \qquad \mu = 0,\ldots,D_0-1. \tag{7.7}$$

With the harmonic gauge constraint (7.7), Eq. (7.1) then reads

$$\sum_{i=2}^{n} \varphi^i_\mu \varphi^i_\nu + \sum_{\alpha\beta=1}^{l} C_{\alpha\beta} b^\alpha_\mu b^\beta_\nu = \sum_{i=1}^{n} d_i b^i_\mu b^i_\nu + \sum_{\alpha\beta=1}^{l} C_{\alpha\beta} b^\alpha_\mu b^\beta_\nu \stackrel{!}{=} 0, \qquad \mu,\nu = 0,\ldots,D_0-1 \ . \tag{7.8}$$

For convenience, one can set $c^A := 0$, $A = 1, \ldots, n+l$. Then $\gamma = 0$, whence the harmonic coordinates are simultaneously proper coordinates, and the solution reads explicitly,

$$g = \eta_{\mu\nu} d\tau^\mu \otimes d\tau^\nu + \sum_{i=1}^{n} e^{2b^i_\lambda \tau^\lambda} g^{(i)}, \tag{7.9}$$

with linear coefficients b^i_μ, $i = 1, \ldots, n$, $\mu = 0, \ldots, D_0 - 1$, satisfying D_0 linear constraints (7.7) (the harmonic gauge) and ${D_0}^2$ quadratic constraints (7.8) (the harmonic Wheeler-de Witt constraints).

This solution shows a generalized inflationary behaviour, which extends the familiar notion of inflation w.r.t. time, as in cosmology, to inflation w.r.t. the internal degrees of freedom on the D_0-dimensional world manifold of an extended object. The constraint (7.7) implies that the total $(D - D_0)$- dimensional volume remains constant (like in a steady state universe [82]) on the world manifold M_0, although here (unlike the stationary case [82]) individual factor spaces may undergo inflationary expansion or contraction in particular directions on M_0. In the standard cosmological case $D_0 = 1$, this solution agrees with the one described in [83].

7.2 Orthobrane solutions with ${}^{(\mathrm{E})}V = 0$

Now we present a class of solutions with ${}^{(\mathrm{E})}V = 0$, where the field equations read

$$R_{\mu\nu}[g^{(0)}] = \hat{G}_{AB} \partial_\mu \sigma^A \partial_\nu \sigma^B + \sum_{s \in S} \varepsilon_s e^{2L_{As}\sigma^A} \partial_\mu \Phi^s \partial_\nu \Phi^s, \qquad \mu, \nu = 1, \ldots, D_0, \tag{7.10}$$

$$\hat{G}_{AB} \triangle[g^{(0)}] \sigma^B - \sum_{s \in S} \varepsilon_s L_{As} e^{2L_{Cs}\sigma^C} (\partial \Phi^s)^2 = 0, \qquad A = 1, \ldots, N, \tag{7.11}$$

$$\partial_\mu \left(\sqrt{|g^{(0)}|} g^{(0)\mu\nu} e^{2L_{As}\sigma^A} \partial_\nu \Phi^s \right) = 0, \qquad s \in S. \tag{7.12}$$

For the Abelian part of the target space metric we set $(\hat{G}^{AB}) := (\hat{G}_{AB})^{-1}$.

$$< X, Y >:= X_A \hat{G}^{AB} X_B. \tag{7.13}$$

For $s \in S$ let us now consider vectors

$$L_s = (L_{As}) \in \mathbb{R}^N. \tag{7.14}$$

Definition: A non-empty set S is called an *orthobrane* index set, iff there exists a family of real non-zero coefficients $\{\nu_s\}_{s \in S}$, such that

$$< L_s, L_r >= (L^T \hat{G}^{-1} L)_{sr} = -\varepsilon_s (\nu_s)^{-2} \delta_{sr}, \qquad s, r \in S. \tag{7.15}$$

For $s \in S$ and $A = 1, \ldots, N$, we set

$$\alpha^A_s := -\varepsilon_s (\nu_s)^2 \hat{G}^{AB} L_{Bs}. \tag{7.16}$$

Here, (7.15) is just an orthogonality condition for the vectors L_s, $s \in S$. Note that

$< L_s, L_s >$ has just the opposite sign of ε_s, $s \in S$. With the definition above, we obtain an existence criterion for solutions.

Theorem: Let S be an orthobrane index set with coefficients (7.16). If for any $s \in S$ there is a function $H_s > 0$ on M_0 such that

$$\triangle[g^{(0)}]H_s = 0, \tag{7.17}$$

i.e. H_s is harmonic on M_0, then, the field configuration

$$R_{\mu\nu}[g^{(0)}] = 0, \qquad \mu, \nu = 1, \ldots, D_0, \tag{7.18}$$

$$\sigma^A = \sum_{s \in S} \alpha_s^A \ln H_s, \qquad A = 1, \ldots, N, \tag{7.19}$$

$$\Phi^s = \frac{\nu_s}{H_s}, \qquad s \in S, \tag{7.20}$$

satisfies the field equations (7.10)-(7.12). □

This theorem follows just from substitution of (7.15)-(7.20) into the equations of motion (7.10)-(7.12). From (6.59), (6.58) and (7.13) we get

$$< L_s, L_r >= G^{ij} l_{iI} l_{jJ} + C^{\alpha\beta} \lambda_{\alpha a} \lambda_{\beta b}, \tag{7.21}$$

with $s = (a, I)$ and $r = (b, J)$ in S ($a, b \in \Delta$, $I \in \Omega_a$, $J \in \Omega_b$). Here, the inverse of the dilatonic midisuperspace metric G_{ij} is given by

$$G^{ij} = \frac{\delta_{ij}}{d_i} + \frac{1}{2 - D}, \tag{7.22}$$

whence, for $I, J \in \Omega$, with topological numbers l_{iI} from (6.57), we obtain

$$G^{ij} l_{iI} l_{jJ} = D(I \cap J) + \frac{D(I)D(J)}{2 - D}, \tag{7.23}$$

which is again a purely topological number.

We set $\nu_{a,I} := \nu_{(a,I)}$. Then, due to (7.21) and (7.23), the orthobrane condition (7.15) reads

$$D(I \cap J) + \frac{D(I)D(J)}{2 - D} + C^{\alpha\beta} \lambda_{\alpha a} \lambda_{\beta b} = -\varepsilon(I)(\nu_{a,I})^{-2} \delta_{ab} \delta_{I,J}, \tag{7.24}$$

for $a, b \in \Delta$, $I \in \Omega_a$, $I \in \Omega_b$. With $(a, I) = s \in S$, the coefficients (7.16) are

$$\alpha_s^i = -\varepsilon(I) G^{ij} l_{jI} \nu_{a,I}^2 = \varepsilon(I) \Big(\sum_{j \in I} \delta_j^i + \frac{D(I)}{2 - D} \Big) \nu_{a,I}^2, \qquad i = 1, \ldots, n, \tag{7.25}$$

$$\alpha_s^\beta = -\varepsilon(I) C^{\beta\gamma} \lambda_{\gamma a} \nu_{a,I}^2, \qquad \beta = 1, \ldots, l \tag{7.26}$$

With $(\sigma^A) = (\phi^i, \varphi^\beta)$, according to (7.19),

$$\beta^i = \sum_{s \in S} \alpha_s^i \ln H_s, \qquad i = 1, \ldots, n, \tag{7.27}$$

$$\phi^\beta = \sum_{s \in S} \alpha_s^\beta \ln H_s, \qquad \beta = 1, \ldots, l, \tag{7.28}$$

and the harmonic gauge reads

$$\gamma = \sum_{s\in S} \alpha_s^0 \ln H_s, \tag{7.29}$$

where

$$\alpha_s^0 := \varepsilon(I)\frac{D(I)}{2-D}\nu_{a,I}^2. \tag{7.30}$$

With $H_{a,I} := H_{(a,I)}$, (7.25), (7.26), and (7.30), the solution of (7.18) - (7.20) reads

$$\begin{aligned} g &= \Big(\prod_{s\in S} H_s^{2\alpha_s^0}\Big) g^{(0)} + \sum_{i=1}^{n}\Big(\prod_{s\in S} H_s^{2\alpha_s^i}\Big) g^{(i)} \\ &= \left(\prod_{(a,I)\in S} H_{a,I}^{\varepsilon(I)2D(I)\nu_{a,I}^2}\right)^{1/(2-D)} \left\{ g^{(0)} + \sum_{i=1}^{n}\left(\prod_{(a,I)\in S, I\ni i} H_{a,I}^{\varepsilon(I)2\nu_{a,I}^2}\right) g^{(i)}\right\}, \\ &\quad \text{with } \mathrm{Ric}[g^{(0)}] = 0, \qquad \mathrm{Ric}[g^{(i)}] = 0, \qquad i = 1, \dots n, \end{aligned} \tag{7.31}$$

$$\phi^\beta = \sum_{s\in S} \alpha_s^\beta \ln H_s = -\sum_{(a,I)\in S} \varepsilon(I) C^{\beta\gamma}\lambda_{\gamma a}\nu_{a,I}^2 \ln H_{a,I}, \qquad \beta = 1, \dots, l, \tag{7.32}$$

$$A^a = \sum_{I\in\Omega_a} \frac{\nu_{a,I}}{H_{a,I}}\tau_I, \qquad a \in \Delta, \tag{7.33}$$

where forms τ_I are defined in (6.16), parameters $\nu_s \neq 0$ and λ_a satisfy the orthobrane condition (7.24), H_s are positive harmonic functions on M_0, and $\mathrm{Ric}[g^{(i)}]$ denotes the Ricci-tensor of $g^{(i)}$. Finally recall that these solutions are subject to the *orthobrane* constraints

$$D(I\cap J) + \frac{D(I)D(J)}{2-D} + C^{\alpha\beta}\lambda_{\alpha a}\lambda_{\beta b} = -\varepsilon(I)(\nu_{a,I})^{-2}\delta_{ab}\delta_{I,J}\ , \quad 0 \neq \nu_{a,I} \in \mathbb{R}\ , \tag{7.34}$$

for $a, b \in \Delta$, $I \in \Omega_a$, $I \in \Omega_b$. These condition lead to specific intersection rules for the p-branes involved. Some concrete examples of *orthobrane* solutions have been elaborated in [31].

For positive definite $(C_{\alpha\beta})$ (or $(C^{\alpha\beta})$) and $D_0 \geq 2$, (7.34) implies

$$\varepsilon(I) = -1, \tag{7.35}$$

for all $I \in \Omega_a$, $a \in \Delta$. Then, the restriction $g|_{M_I}$ of the metric (7.31) to a membrane manifold M_I has an odd number of negative eigenvalues, i.e. linearly independent time-like directions. However, if the metric $(C_{\alpha\beta})$ in the space of scalar fields is not positive definite, then (7.35) may be violated for sufficiently negative $C^{\alpha\beta}\lambda_{\alpha a}\lambda_{\beta b} < 0$.

7.3 Spherically symmetric p-branes

Let us now examine static, spherically symmetric, multidimensional space-times with

$$M = M_{-1}\times M_0 \times M_1 \times \cdots \times M_N, \qquad \dim M_i = d_i, \quad i = 0, \dots, N, \tag{7.36}$$

where $M_{-1} \subset \mathbb{R}$ corresponds to a radial coordinate u, $M_0 = S^2$ is a 2-sphere, $M_1 \subset \mathbb{R}$ is time, and $M_i,\ i > 1$ are internal factor spaces. The metric is assumed correspondingly to be

$$
\begin{aligned}
ds^2 &= \mathrm{e}^{2\alpha(u)} du^2 + \sum_{i=0}^{N} \mathrm{e}^{2\beta_i(u)} ds_i^2 \\
&\equiv -\mathrm{e}^{2\gamma(u)} dt^2 + \mathrm{e}^{2\alpha(u)} du^2 + \mathrm{e}^{2\beta_0(u)} d\Omega^2 + \sum_{i=2}^{N} \mathrm{e}^{2\beta_i(u)} ds_i^2,
\end{aligned}
\tag{7.37}
$$

where $ds_0^2 \equiv d\Omega^2 = d\theta + \sin^2\theta\, d\phi^2$ is the line element on S^2, $ds_1^2 \equiv -dt^2$ with $\beta_1 =: \gamma$, and ds_i^2, $i > 1$, are u-independent line elements of internal Ricci-flat spaces of arbitrary dimensions d_i and signatures ε_i.

For simplicity here let us only consider a single scalar field denoted as φ.

An electric-type $p+2$-form $F_{\mathrm{e}I}$ has a domain given by a product manifold

$$
M_I = M_{i_1} \times \cdots \times M_{i_k}, \tag{7.38}
$$

where

$$
I = \{i_1, \ldots, i_k\} \subset I_0 \stackrel{\text{def}}{=} \{0, 1, \ldots, N\}. \tag{7.39}
$$

The corresponding dimensions are

$$
d(I) \stackrel{\text{def}}{=} \sum_{i \in I} d_i, \qquad\qquad d(I_0) = D - 1. \tag{7.40}
$$

A magnetic-type F-form of arbitrary rank k may be defined as a form on a domain $M_{\overline{I}}$ with $\overline{I} \stackrel{\text{def}}{=} I_0 - I$, dual to an electric-type form,

$$
F_{\mathrm{m}I,\, M_1 \ldots M_k} = \mathrm{e}^{-2\lambda\varphi} (*F)_{\mathrm{e}I,\, M_1 \ldots M_k} \equiv \mathrm{e}^{-2\lambda\varphi} \frac{\sqrt{g}}{k!} \varepsilon_{M_1 \ldots M_k N_1 \ldots N_{D-k}} F_{\mathrm{e}I}^{N_1 \ldots N_{D-k}}, \tag{7.41}
$$

where $*$ is the Hodge operator and ε is the totally antisymmetric Levi-Civita symbol.

For simplicity let us now consider just a single n-form, i.e. a single electric type and a single dual magnetic component, whence

$$
\operatorname{rank} F_{\mathrm{m}I} = D - \operatorname{rank} F_{\mathrm{e}I} = d(\overline{I}), \tag{7.42}
$$

whence $k = n$ in (7.41) and

$$
d(I) = n - 1 \quad \text{for} \quad F_{\mathrm{e}I}, \qquad d(I) = d(I_0) - n = D - n - 1 \quad \text{for} \quad F_{\mathrm{m}I}. \tag{7.43}
$$

All fields must be compatible with spherical symmetry and staticity. Correspondingly, the vector φ of scalars and the $p+2$-forms valued fields depend (besides on their domain as forms) on the radial variable u only.

Furthermore, the domain of the electric form $F_{\mathrm{e}I}$ does not include the sphere $M_0 = S^2$, and $F_{\mathrm{e}I}$ is specified by a u-dependent potential form,

$$
F_{\mathrm{e}I,\, uL_2 \ldots L_n} = \partial_{[u} U_{L_2 \ldots L_n]} \qquad U = U_{L_2, \ldots, L_n} dx^{L_2} \wedge \ldots \wedge dx^{L_n}. \tag{7.44}
$$

Since the time manifold M_1 is a factor space of M_I, the form (7.44) describes an electric $(n-2)$-brane in the remaining subspace of M_I. Similarly (7.41) describes a magnetic $(D-n-2)$-brane in M_I.

Let us label all nontrivial components of F by a collective index $s = (I_s, \chi_s)$, where $I = I_s \subset I_0$ characterizes the subspace of M as described above and $\chi_s = \pm 1$ according to the rule

$$\mathrm{e} \mapsto \chi_s = +1, \qquad\qquad \mathrm{m} \mapsto \chi_s = -1. \tag{7.45}$$

If $1 \in I$, the corresponding p-brane evolves with t and we have a true electric or magnetic field, otherwise the potential (7.44) does not depend on $\overline{M}_0$, i.e. it is just a scalar in 4 dimensions. In this case we call the corresponding electric-type F component (7.44) *electric quasiscalar* and its dual, magnetic-type, F component (7.41) *magnetic quasiscalar*. In general there are four types of F-field components (summarized in Table 1): electric (E), magnetic (M), electric quasiscalar (EQ), magnetic quasiscalar (MQ). The choice

Table 1: Different types of antisymmetric $p+2$-form fields

E	electric ($1 \in I$)	$F_{tuA_3 \ldots A_n}$	A_k (coordinate) index of M_I
M	magnetic ($1 \in I$)	$F_{\theta\phi B_3 \ldots B_n}$	B_l (coordinate) index of $M_{\overline{I}}$
EQ	electric quasiscalar ($1 \notin I$)	$F_{uA_2 \ldots A_n}$	A_k (coordinate) index of M_I
MQ	magnetic quasiscalar ($1 \notin I$)	$F_{t\theta\phi B_4 \ldots B_n}$	B_l (coordinate) index of $M_{\overline{I}}$

of subsets I_s is only constrained by the multidimensional decomposition condition (6.52) for the energy-momentum tensor. Since antisymmetric $p+2$-form field components of type E and M (and type EQ and MQ respectively) just complement each other, they should be considered as independent of each other. In the following we consider all F_s as independent fields (up to index permutations) each with a single nonzero component.

Let us assume Ricci-flat internal spaces. With spherical symmetry and staticity all field become independent of M_0 and M_1 respectively. And the variation reduces further from $\overline{M}_0$ to the radial manifold M_{-1}.

The reparametrization gauge on the lower dimensional manifold here is chosen as the (generalized) harmonic one [32]. Since M_{-1} is 1-dimensional u is a harmonic coordinate, $\Box\, u = 0$, such that

$$\alpha(u) = \sigma_0(u). \tag{7.46}$$

The nonzero Ricci tensor components are

$$\begin{aligned} \mathrm{e}^{2\alpha} R^t_t &= -\gamma'', \\ \mathrm{e}^{2\alpha} R^u_u &= -\alpha'' + \alpha'^2 - \gamma'^2 - 2\beta'^2 - \sum_{i=2}^{N} d_i \beta_i'^2, \\ \mathrm{e}^{2\alpha} R^\theta_\theta &= \mathrm{e}^{2\alpha} R^\phi_\phi = \mathrm{e}^{2\alpha - 2\beta} - \beta'', \end{aligned}$$

$$\mathrm{e}^{2\alpha} R^{b_i}_{a_j} = -\delta^{b_i}_{a_j} \beta''_i \qquad (i, j = 1, \ldots, N), \tag{7.47}$$

where a prime denotes d/du and the indices $a_i,\ b_i$ belong to the i-th internal factor space. The Einstein tensor component G_1^1 does not contain second-order derivatives:

$$\mathrm{e}^{2\alpha} G_1^1 = -\mathrm{e}^{2\alpha - 2\beta} + \frac{1}{2}\alpha'^2 - \frac{1}{2}\Big(\gamma'^2 + 2\beta'^2 + \sum_{i=2}^{N} d_i \beta_i'^2\Big). \tag{7.48}$$

The corresponding component of the Einstein equations is an integral of other components, similar to the energy integral in cosmology.

The generalized Maxwell equations give

$$F^{uM_2\ldots M_n}_{\mathrm{e}I} = Q_{\mathrm{e}I} \mathrm{e}^{-2\alpha - 2\lambda\varphi}, \qquad Q_{\mathrm{e}I} = \mathrm{const}, \tag{7.49}$$

$$F_{\mathrm{m}I,\, uM_1\ldots M_{d(\bar I)}} = Q_{\mathrm{m}I} \sqrt{|g_{\bar I}|}, \qquad Q_{\mathrm{m}I} = \mathrm{const}, \tag{7.50}$$

where $|g_{\bar I}|$ is the determinant of the u-independent part of the metric of $M_{\bar I}$ and Q_s are charges. These solutions provide then the energy momentum tensors, of the electric and magnetic $p+2$-forms written in matrix form,

$$\begin{aligned}
\mathrm{e}^{2\alpha}(T^N_M[F_{\mathrm{e}I}]) &= -\frac{1}{2}\eta_F \varepsilon(I) Q^2_{\mathrm{e}I} \mathrm{e}^{2y_{\mathrm{e}I}} \mathrm{diag}(+1,\ [1]_I,\ [-1]_{\bar I});\\
\mathrm{e}^{2\alpha}(T^N_M[F_{\mathrm{m}I}]) &= \frac{1}{2}\eta_F \varepsilon(\bar I) Q^2_{\mathrm{m}I} \mathrm{e}^{2y_{\mathrm{m}I}} \mathrm{diag}(1,\ [1]_I,\ [-1]_{\bar I}),
\end{aligned} \tag{7.51}$$

where the first position belongs to u and f operating over M_J is denoted by $[f]_J$. The functions $y_s(u)$ are

$$y_s(u) = \sigma(I_s) - \chi_s \lambda\varphi. \tag{7.52}$$

The scalar field contribution to the energy momentum tensor (EMT) is

$$\mathrm{e}^{2\alpha} T^N_M[\varphi] = \frac{1}{2}(\varphi^a)'^2 \mathrm{diag}(+1,\ [-1]_{I_0}). \tag{7.53}$$

The sets $I_s \in I_0$ may be classified by types E, M, EQ, MQ according to the description in the previous section. Denoting I_s for the respective types by $I_{\mathrm{E}},\ I_{\mathrm{M}},\ I_{\mathrm{EQ}},\ I_{\mathrm{MQ}}$, we see from (7.51) that, positive electric and magnetic energy densities require

$$\eta_F = -\varepsilon(I_{\mathrm{E}}) = \varepsilon(\bar I_{\mathrm{M}}) = \varepsilon(I_{\mathrm{EQ}}) = -\varepsilon(\bar I_{\mathrm{MQ}}). \tag{7.54}$$

If t is the only time coordinate, (7.54) with $\eta_F = 1$ holds for any choices of I_s. If there exist other times, then the relations (7.54) constrain the subspaces where the different F components may be specified.

Since the total EMT on the r.h.s. of the Einstein equations has the property

$$T^u_u + T^\theta_\theta = 0, \tag{7.55}$$

the corresponding combination on the l.h.s becomes an integrable Liouville form

$$\begin{aligned}
G^u_u + G^\theta_\theta &= \mathrm{e}^{-2\alpha}[-\alpha'' + \beta_0'' + \mathrm{e}^{2\alpha - 2\beta_0}] = 0,\\
\mathrm{e}^{\beta_0 - \alpha} &= s(k, u),
\end{aligned} \tag{7.56}$$

where k is an integration constant (IC) and the function $s(k,\cdot)$ is defined as follows:

$$s(k,u) \stackrel{\rm def}{=} \begin{cases} k^{-1}\sinh ku, & k>0 \\ u, & k=0 \\ k^{-1}\sin ku, & k<0 \end{cases} \tag{7.57}$$

Another IC is suppressed by adjusting the origin of the u coordinate.

With (7.56) the D-dimensional line element may be written in the form

$$ds^2 = \frac{{\rm e}^{-2\sigma_1}}{s^2(k,u)}\biggl[\frac{du^2}{s^2(k,u)} + d\Omega^2\biggr] + \sum_{i=1}^N {\rm e}^{2\beta_i} ds_i^2 \tag{7.58}$$

where σ_1 has been defined in (6.8).

Let us treat the whole set of unknowns $\beta_i(u)$, $\varphi(u)$ as a real-valued vector function $x(u)$ in an $(N{+}1)$-dimensional vector space V, with components $x^A = \beta_A$ for $A = 1,\ldots,N$ and $x^{N+1} = \varphi$.

Then the field equations for β_i and φ coincide with the equations of motion corresponding to the Lagrangian of a Euclidean Toda-like system

$$L = \overline{G}_{AB} {x'}^A {x'}^B - V_Q(y), \qquad V_Q(y) = \sum_s \theta_s Q_s^2 {\rm e}^{2y_s}, \tag{7.59}$$

where, according to (7.54), $\theta_s = 1$ if F_s is a true electric or magnetic field and $\theta_s = -1$ if F_s is quasiscalar. The nondegenerate, symmetric matrix

$$(\overline{G}_{AB}) = \begin{pmatrix} G_{ij} & 0 \\ 0 & 1 \end{pmatrix}, \qquad G_{ij} = d_i d_j + d_i \delta_{ij} \tag{7.60}$$

defines a positive-definite metric in V. The energy constraint corresponding to (7.59) is

$$E = {\sigma'_1}^2 + \sum_{i=1}^N d_i {\beta'}_i^2 + {\varphi'}^2 + V_Q(y) = \overline{G}_{AB} {x'}^A {x'}^B + V_Q(y) = 2k^2 {\rm sign} k, \tag{7.61}$$

with k from (7.56). The integral (7.61) follows here from the (uu)-component of (6.45).

The functions $y_s(u)$ (7.52) can be represented as scalar products in V (recall that $s = (I_s, \chi_s)$):

$$y_s(u) = Y_{s,A} x^A, \qquad (Y_{s,A}) = (d_i \delta_{iI_s}, \quad -\chi_s\lambda), \tag{7.62}$$

where $\delta_{iI} := \sum_{j\in I}\delta_{ij}$ is an indicator for i belonging to I (1 if $i\in I$ and 0 otherwise).

The contravariant components of Y_s are found using the matrix $\overline{G}^{AB}$ inverse to $\overline{G}_{AB}$:

$$(\overline{G}^{AB}) = \begin{pmatrix} G^{ij} & 0 \\ 0 & 1 \end{pmatrix}, \qquad G^{ij} = \frac{\delta^{ij}}{d_i} - \frac{1}{D-2} \tag{7.63}$$

$$({Y_s}^A) = \biggl(\delta_{iI_s} - \frac{d(I_s)}{D-2}, \quad -\chi_s\lambda\biggr), \tag{7.64}$$

and the scalar products of different Y_s, whose values are of primary importance for the integrability of our system, are

$$Y_{s,A} {Y_{s'}}^A = d(I_s \cap I_{s'}) - \frac{d(I_s)d(I_{s'})}{D-2} + \chi_s\chi_{s'}\lambda^2. \tag{7.65}$$

7.4 Black holes with EM branes

In [27] it was shown that quasiscalar components of the F-fields are incompatible with orthobrane black holes. Therefore let us now consider only two F-field components, Type E and Type M according to the classification above. They will be electric as $F_{\rm e}$ and $F_{\rm m}$ and the corresponding sets $I_s \subset I_0$ as $I_{\rm e}$ and $I_{\rm m}$. Then a minimal configuration (7.36) of the manifold M compatible with an arbitrary choice of I_s has the following form:

$$N = 5, \qquad I_0 = \{0,1,2,3,4,5\}, \qquad I_e = \{1,2,3\}, \qquad I_m = \{1,2,4\}, \tag{7.66}$$

so that

$$\begin{aligned} &d(I_0) = D-1, \quad d(I_{\rm e}) = n-1, \quad d(I_{\rm m}) = D-n-1, \quad d(I_{\rm e}\cap I_{\rm m}) = 1+d_2; \\ &d_1 = 1, \qquad\qquad d_2+d_3 = d_3+d_5 = n-2. \end{aligned} \tag{7.67}$$

The relations (7.67) show that, given D and d_2, all d_i are known.

This corresponds to an electric $(n-2)$-brane located on the subspace $M_2 \times M_3$ and a magnetic $(D-n-2)$-brane on the subspace $M_2 \times M_4$. Their intersection dimension $d_{\rm int} = d_2$ turns out to determine qualitative properties of the solutions.

The index s now takes the two values e and m and

$$\begin{aligned} Y_{{\rm e},A} &= (1, d_2, d_3, 0, 0, -\lambda); \\ Y_{{\rm m},A} &= (1, d_2, 0, d_4, 0, \lambda); \\ Y_{\rm e}^A &= (1,1,1,0,0,-\lambda) - \frac{n-1}{D-2}(1,1,1,1,1,0); \\ Y_{\rm m}^A &= (1,1,0,1,0,\lambda) - \frac{D-n-1}{D-2}(1,1,1,1,1,0), \end{aligned} \tag{7.68}$$

where the last component of each vector refers to $x^{N+1} = x^6 = \varphi$.

In the solutions presented below the set of ICs will be reduced by the condition that the space-time be asymptotically flat at spatial infinity ($u = 0$) and by a choice of scales in the relevant directions. Namely, we put

$$\beta_i(0) = 0 = \varphi(0) \qquad i = 1,2,3,4,5. \tag{7.69}$$

The requirement $\varphi(0) = 0$ is convenient and may be always satisfied by a redefinition of the charges. The conditions $\beta_i(0) = 0$ ($i > 1$) mean that the real scales of the extra dimensions are hidden in the internal metrics ds_i^2 independent of whether or not they are assumed to be compact.

In the following, both cases, orthobrane solutions and solutions with degenerate charges, are considered first generally and then for the minimal configuration (7.66)-(7.69).

7.4.1 Solutions with orthobranes

Assuming that the vectors Y_s are mutually orthogonal with respect to the metric $\overline{G}_{AB}$, i.e.

$$Y_{s,A} Y_{s'}{}^{A} = \delta_{ss'} N_s^2, \tag{7.70}$$

the number of functions y_s does not exceed the number of equations, and the system becomes integrable. Due to (7.43), the norms N_s are actually s-independent:

$$N_s^2 = d(I_s)\Big[1 - \frac{d(I_s)}{D-2}\Big] + \lambda^2 = \frac{(n-1)(D-n-1)}{D-2} + \lambda^2 \stackrel{\rm def}{=} \frac{1}{\nu}, \tag{7.71}$$

$\nu > 0$.

Due to (7.70), the functions $y_s(u)$ obey the decoupled equations

$$y_s'' = \theta_s \frac{Q_s^2}{\nu} e^{2y_s}, \tag{7.72}$$

whence

$$e^{-y_s(u)} = \begin{cases} (|Q_s|/\sqrt{\nu}) s(h_s,\ u+u_s), & \theta = +1, \\ [|Q_s|/(\sqrt{\nu}h_s)] \cosh[h_s(u+u_s)], \qquad h_s > 0, & \theta = -1. \end{cases} \tag{7.73}$$

where h_s and u_s are ICs and the function s was defined in (7.57). For the functions $x^A(u)$ we obtain:

$$x^A(u) = \nu \sum_s Y_s{}^A y_s(u) + c^A u + \bar{c}^A, \tag{7.74}$$

where the vectors of ICs c^A and $\bar{c}^A$ satisfy the orthogonality relations $c^A Y_{s,A} = \bar{c}^A Y_{s,A} = 0$, or

$$c^i d_i \delta_{iI_s} - \lambda c^{N+1} \chi_s = 0, \qquad \bar{c}^i d_i \delta_{iI_s} - \lambda \bar{c}^{N+1} \chi_s = 0. \tag{7.75}$$

Specifically, the logarithms of the scale factors β_i and the scalar field φ are

$$\beta_i(u) = \nu \sum_s \Big[\delta_{iI_s} - \frac{d(I_s)}{D-2}\Big] y_s(u) + c^i u + \bar{c}^i, \tag{7.76}$$

$$\varphi(u) = -\lambda\nu \sum_s y_s(u) + c^{N+1} u + \bar{c}^{N+1}, \tag{7.77}$$

and the function σ_1 which appears in the metric (7.58) is

$$\sigma_1 = -\frac{\nu}{D-2} \sum_s d(I_s)\, y_s(u) + c^0 u + \bar{c}^0 \tag{7.78}$$

with

$$c^0 = \sum_{i=1}^{N} d_i c^i, \qquad \bar{c}^0 = \sum_{i=1}^{N} d_i \bar{c}^i. \tag{7.79}$$

Finally, (7.61) now reads

$$E = \nu \sum_s h_s^2 \mathrm{sign} h_s + \overline{G}_{AB} c^A c^B = 2k^2 \mathrm{sign} k. \tag{7.80}$$

The relations (7.46), (7.49), (7.50), (7.56), (7.58), (7.73)–(7.80), along with the definitions (7.57) and (7.71) and the restriction (7.70), entirely determine the general solution.

For the minimal configuration (7.66)-(7.69), the orthogonality condition (7.70) reads

$$\lambda^2 = d_2 + 1 - \frac{1}{D-2}(n-1)(D-n-1) \tag{7.81}$$

In particular, in dilaton gravity $n = 2,\ d_2 = 0$ and the integrability condition (7.81) just reads $\lambda^2 = 1/(D-2)$, which is a well-known relation of string gravity. The familiar Reissner-Nordström solution, $D = 4,\ n = 2,\ \lambda = 0,\ d_2 = 0$ does *not* satisfy Eq. (7.81). (It will be recovered indeed as a degenerate case below.) Some examples of configurations satisfying the orthogonality condition (7.81) in the purely topological case $\lambda = 0$ are summarized in Table 2 (including the values of the constants B and C from (7.93)). In this case (7.81) is just a Diophantus equation for D, n and d_2.

Table 2: Orthobrane solutions with $\lambda = 0$

	n	$d(I_{\rm e})$	$d(I_{\rm m})$	d_2	B	C
$D = 4m+2$ $(m \in \mathbb{N})$	2m+1	2m	2m	m-1	1/m	1/m
D= 11	4	3	6	1	2/3	1/3
	7	6	3	1	1/3	2/3

The solution is entirely determined by inserting (7.68) into (7.74) with $\bar{c}^A = 0$ due to (7.69),

$$x^A(u) = \nu \sum_s Y_s{}^A y_s(u) + c^A u; \qquad {\rm e}^{-y_s(u)} = (|Q_s|/\sqrt{\nu}) s(h_s,\ u + u_s). \tag{7.82}$$

Due to (7.81) the parameter ν is

$$\nu = 1/\sqrt{1+d_2}. \tag{7.83}$$

The constants are connected by the relations

$$\begin{aligned} &(|Q_{\rm e,m}|/\nu)\, s(h_{\rm e,m}, u_{\rm e,m}) = 1; \\ &c^1 + d_2 c^2 + d_3 c^3 - \lambda c^6 = 0; \qquad\qquad c^1 + d_2 c^2 + d_4 c^4 + \lambda c^6 = 0; \\ &\frac{h_{\rm e}^2 {\rm sign} h_{\rm e} + h_{\rm m}^2 {\rm sign} h_{\rm m}}{1+d_2} + G_{ij} c^i c^j + (c^6)^2 = 2k^2 {\rm sign} k, \end{aligned} \tag{7.84}$$

where the matrix G_{ij} is given in (7.60) and all $\bar{c}^A = 0$ due to the boundary conditions (7.69). The fields φ and F are given by Eqs. (7.49), (7.50), (7.77).

This solution contains 8 nontrivial, independent ICs, namely, $Q_{\rm e}, Q_{\rm m}, h_{\rm e}, h_{\rm m}$ and 4 others from the set c^A constrained by (7.84).

For black holes, we require that all $|\beta_i| < \infty$, $i = 2, \ldots, N$ (regularity of extra dimensions), $|\varphi| < \infty$ (regularity of the scalar field) and $|\beta_0| < \infty$ (finiteness of the spherical radius) as $u \to \infty$. With $y_s(u) \sim -h_s u$, this leads to the following constraints on the ICs:

$$c^A = -k \sum_s \Big(\delta_{1I_s} + \nu Y_s{}^A h_s\Big), \tag{7.85}$$

where $A = 1$ corresponds to $i = 1$. Via orthonormality relations (7.75) for c^A, we obtain

$$h_s = k\delta_{1I_s}, \tag{7.86}$$

$$c^A = -k\delta_1^A + k\nu \sum_s \delta_{1I_s} {Y_s}^A, \tag{7.87}$$

and (7.80) then holds automatically.

Let us now consider the case where (7.86) and (7.87) with $\delta_{1I_s} = 1$ hold. After a transformation $u \mapsto R$, to isotropic coordinates given by the relation

$$\mathrm{e}^{-2ku} = 1 - 2k/R, \tag{7.88}$$

we obtain

$$ds^2 = -\frac{1-2k/R}{P_\mathrm{e}^B P_\mathrm{m}^C} dt^2 + P_\mathrm{e}^C P_m^B \left(\frac{dR^2}{1-2k/R} + R^2 d\Omega^2 \right) + \sum_{i=2}^{5} \mathrm{e}^{2\beta_i(u)} ds_i^2, \tag{7.89}$$

$$\mathrm{e}^{2\beta_2} = {P_\mathrm{e}}^{-B} {P_\mathrm{m}}^{-C}, \qquad \mathrm{e}^{2\beta_3} = (P_\mathrm{m}/P_\mathrm{e})^B,$$

$$\mathrm{e}^{2\beta_4} = (P_\mathrm{e}/P_\mathrm{m})^C, \qquad \mathrm{e}^{2\beta_5} = {P_\mathrm{e}}^C {P_\mathrm{m}}^B, \tag{7.90}$$

$$e^{2\lambda\varphi} = (P_\mathrm{e}/P_\mathrm{m})^{2\lambda^2/(1+d_2)}, \tag{7.91}$$

$$F_{01M_3\ldots M_n} = -Q_\mathrm{e}/(R^2 P_\mathrm{e}), \qquad F_{23M_3\ldots M_n} = Q_\mathrm{m} \sin\theta, \tag{7.92}$$

with the notations

$$P_\mathrm{e,m} = 1 + p_\mathrm{e,m}/R, \qquad p_\mathrm{e,m} = \sqrt{k^2 + (1+d_2)Q^2_\mathrm{e,m}} - k;$$

$$B = \frac{2(D-n-1)}{(D-2)(1+d_2)}, \qquad C = \frac{2(n-1)}{(D-2)(1+d_2)}. \tag{7.93}$$

The BH gravitational mass as determined from a comparison of (7.89) with the Schwarzschild metric for $R \to \infty$ is

$$G_N M = k + \frac{1}{2}(Bp_\mathrm{e} + Cp_\mathrm{m}), \tag{7.94}$$

where G_N is the Newtonian gravitational constant. This expression, due to $k > 0$, provides a restriction upon the charge combination for a given mass, namely,

$$B|Q_\mathrm{e}| + C|Q_\mathrm{m}| < 2G_N M/\sqrt{1+d_2}. \tag{7.95}$$

The inequality is replaced by equality in the extreme limit $k = 0$. For $k = 0$ our BH turns into a naked singularity (at the centre $R = 0$) for any $d_2 > 0$, while for $d_2 = 0$ the zero value of R is not a centre ($g_{22} \neq 0$) but a horizon. In the latter case, if $|Q_e|$ and $|Q_m|$ are different, the remaining extra-dimensional scale factors are smooth functions for all $R \geq 0$.

For a static, spherical BH one can define a Hawking temperature $T_H := \kappa/2\pi$ as given by the surface gravity κ. With a generalized Komar integral (see e.g. [84])

$$M(r) := -\frac{1}{8\pi} \int_{S_r} *d\xi \tag{7.96}$$

over the time-like Killing form ξ, the surface gravity can be evaluated as

$$\kappa = M(r_H)/(r_H)^2 = (\sqrt{|g_{00}|})' \Big/ \sqrt{g_{11}}\Big|_{r=r_H} = \mathrm{e}^{\gamma-\alpha}|\gamma'|\,\Big|_{r=r_H}, \tag{7.97}$$

where a prime, α, and γ are understood in the sense of the general metric (7.37) and k_{B} is the Boltzmann constant. The expression (7.97) is invariant with respect to radial coordinate reparametrization, as is necessary for any quantity having a direct physical meaning. It is also invariant under conformal mappings with a conformal factor which is smooth at the horizon.

Substituting g_{00} and g_{11} from (7.89), one obtains:

$$T_H = \frac{1}{2\pi k_{\mathrm{B}}}\frac{1}{4k}\left[\frac{4k^2}{(2k+p_{\mathrm{e}})(2k+p_{\mathrm{m}})}\right]^{1/(d_2+1)}. \tag{7.98}$$

If $d_2 = 0$ and both charges are nonzero, this temperature tends to zero in the extreme limit $k \to 0$; if $d_2 = 1$ and both charges are nonzero, it tends to a finite limit, and in all other cases it tends to infinity. Remarkably, it is determined by the p-brane intersection dimension d_2 rather than the whole space-time dimension D.

7.4.2 Solutions with degenerate brane charges $Q_{\mathrm{e}}^2 = Q_{\mathrm{m}}^2$

In this degenerate case, solutions can be found which need not satisfy the orthobrane condition (7.70). Let us suppose that two functions (7.52), say, y_1 and y_2, coincide up to an addition of a constant (which may be then absorbed by re-defining a charge Q_1 or Q_2) while corresponding vectors Y_1 and Y_2 are neither coinciding, nor orthogonal (otherwise we would have the previously considered situation). Substituting $y_1 \equiv y_2$ into (7.62), one obtains

$$(Y_{1,A} - Y_{2,A})x^A = 0. \tag{7.99}$$

This is a constraint reducing the number of independent unknowns x^A. Furthermore, substituting (7.99) to the Lagrange equations for x^A,

$$-(Y_{1,A} - Y_{2,A})x''^A = \sum_s \theta_s Q_s^2 \mathrm{e}^{2y_s} Y_s^A (Y_{1,A} - Y_{2,A}) = 0. \tag{7.100}$$

In this sum all coefficients of different functions e^{2y_s} must be zero. This yields new orthogonality conditions

$$Y_s^A (Y_{1,A} - Y_{2,A}) = 0, \qquad s \neq 1, 2, \tag{7.101}$$

now for the difference $Y_1 - Y_2$ and other Y_s, and with Eq. (7.71) the relation

$$(\nu^{-1} - Y_1^A Y_{2,A})(\theta_1 Q_1^2 - \theta_2 Q_2^2) = 0. \tag{7.102}$$

The first multiplier in (7.102) is positive ($\overline{G}_{AB}$ is positive-definite, hence a scalar product of two different vectors with equal norms is smaller than their norm squared). Therefore

$$\theta_1 = \theta_2, \qquad Q_1^2 = Q_2^2. \tag{7.103}$$

Imposing the constraints (7.99), (7.101), (7.103), reduces the numbers of unknowns and integration constants, and simultaneously also reduces the number of restrictions on the input parameters (by the orthogonality conditions (7.70)). Due to (7.103), this is only possible when the two components with coinciding charges are of equal nature: both must be either true electric/magnetic ones ($\theta_s = 1$), or quasiscalar ones ($\theta_s = -1$). Correspondingly, we now set $y(u) := y_{\rm e} = y_{\rm m}$ and $Q^2 := Q^2_{\rm e} = Q^2_{\rm m}$.

For the minimal configuration (7.66)–(7.68), eq. (7.99) yields

$$d_3\beta_3 - d_4\beta_4 - 2\lambda\varphi = 0. \tag{7.104}$$

Eqs. (7.101) are irrelevant here since we are dealing with two functions y_s only. The equations of motion for x^A now take the form

$$x^{A\prime\prime} = Q^2 {\rm e}^{2y}(Y^A_{\rm e} + Y^A_{\rm m}). \tag{7.105}$$

Their proper combination gives $y'' = (1+d_2)Q^2{\rm e}^{2y}$, whence

$${\rm e}^{-y} = \sqrt{(1+d_2)Q^2}\, s(h, u+u_1) \tag{7.106}$$

where the function s is defined in (7.57) and h, u_1 are ICs and, due to (7.69), $\sqrt{(1+d_2)Q^2}\, s(h,u_1) = 1$. Other unknowns are easily determined using (7.105) and (7.69):

$$\begin{aligned} x^A &= \nu Y^A y + c^A; \qquad Y^A = Y^A_{\rm e} + Y^A_{\rm m} = (1,1,0,0,-1,0); \\ \sigma_1 &= -\nu y + c_0 u. \end{aligned} \tag{7.107}$$

Here, as in (7.83), $\nu = 1/(1+d_2)$, but it is now just a notation. The constants $c_0,\ h,\ c^A$ ($A = 1,\ldots,6$) and k (see (7.56)) are related by

$$\begin{aligned} &-c^0 + \sum_{i=1}^{5} d_i c^i = 0, \qquad c^1 + d_2c^2 + d_3c^3 - \lambda c^6 = 0, \qquad c^1 + d_2c^2 + d_4c^4 + \lambda c^6 = 0, \\ &2k^2 {\rm sign} k = \frac{2h^2 {\rm sign} h}{1+d_2}(c^0)^2 + \sum_{i=1}^{5} d_i (c^i)^2 + (c^6)^2. \end{aligned} \tag{7.108}$$

Extra-dimensional scale factors remain finite as $u \to u_{max}$ in the case of a BH. It is specified by the following values of the ICs:

$$k = h > 0, \quad c^3 = c^4 = c^6 = 0, \quad c_2 = -c_5 = -\frac{k}{1+d_2}\,, \quad c_0 = c^1 = -\frac{d_2 k}{1+d_2}\,. \tag{7.109}$$

The event horizon occurs at $u = \infty$. After the same transformation (7.88) the metric takes the form

$$\begin{aligned} ds^2_D = &-\frac{1-2k/R}{(1+p/R)^{2\nu}}dt^2 + (1+p/R)^{2\nu}\left(\frac{dR^2}{1-2k/R} + R^2 d\Omega^2\right) \\ &+(1+p/R)^{-2\nu}ds_2^2 + ds_3^2 + ds_4^2 + (1+p/R)^{2\nu}ds_5^2 \end{aligned} \tag{7.110}$$

with the notation

$$p = \sqrt{k^2 + (1+d_2)Q^2} - k. \tag{7.111}$$

The fields φ and F are determined by the relations

$$\varphi \equiv 0\ , \qquad F_{01L_3\ldots L_n} = -\frac{Q}{R^2(1+p/R)}, \qquad F_{23L_3\ldots L_n} = Q\sin\theta. \tag{7.112}$$

$$G_N M = k + p/(1+d_2), \tag{7.113}$$

The Hawking temperature can be calculated as before,

$$T = \frac{1}{2\pi k_{\rm B}}\frac{1}{4k}\left(\frac{2k}{2k+p}\right)^{2/(d_2+1)}. \tag{7.114}$$

The well-known results for the Reissner-Nordström metric are recovered when $d_2 = 0$. In this case $T \to 0$ in the extreme limit $k \to 0$. For $d_2 = 1$, T tends to a finite limit as $k \to 0$ and for $d_2 > 1$ it tends to infinity. As is the case with two different charges, T does not depend on the space-time dimension D, but depends on the p-brane intersection dimension d_2.

7.5 Spatially homogeneous solutions

For dimension $D_0 = 1$, the σ-model action has been derived previously in [38], [39]. In this case the harmonic gauge reads $f = -\gamma + \sum_{i=1}^n d_i\beta^i$, whence in the action the kinetic $(\partial\gamma)^2$ terms cancel. Thus, with $g^{(0)} = -dt\otimes dt$ the action describes (according to [38], Eqs. (6.9)-(6.15) in the matter free case) the form of a classical massive particle in minisuperspace, i.e.

$$S = \int L^{(f)}N dt, \qquad L = \{\frac{m}{2}\ {}^{(f)}N^{-2}\ G_{ij}\dot\beta^i\dot\beta^j - V(\beta^i)\} \tag{7.115}$$

where $G_{ij} = d_i\delta_{ij} - d_i d_j$,

$$V((\beta^i)) = -\frac{m}{2}e^{2d_j\beta^j}\sum_{i=1}^n e^{-2\beta^i}R[g^{(i)}], \tag{7.116}$$

and ${}^fN = e^{-f}$ gauges the lapse of time.

The D-dimensional space-time manifold M may be the product of an interval of the time axis R and n manifolds $M_1,\ldots,M_n$, i.e.

$$M = R\times M_1\times\ldots\times M_n. \tag{7.117}$$

The product of some of the manifolds $M_1,\ldots,M_k$, $1\le k\le 3$, gives the external 3-dimensional space and the remaining part $M_{k+1},\ldots,M_n$ stands for so-called internal spaces. We suppose that the internal spaces are compact, however the models with noncompact internal spaces are also discussed in [85], [86], [16], [32].

The manifold M is equipped with the metric

$$\mathrm{g} = -\mathrm{e}^{2\gamma(t)}\mathrm{d}t\otimes\mathrm{d}t + \sum_{i=1}^n \exp[2\beta^i(t)]\mathrm{g}^{(i)}, \tag{7.118}$$

where $\gamma(t)$ is an arbitrary function determining the time t and $\mathrm{g}^{(i)}$ is the metric on the manifold M_i. Models of such type were considered previously by a number of authors for different sources: vacuum [87], [88], [89], [90]; minimally coupled scalar field [91]; perfect fluid [92], [93], [94], [95], [96], [97]; viscous fluid [98].

We assume that the manifolds $M_1, \ldots, M_n$ are Ricci-flat, i.e. the components of the Ricci tensor for the metrics $\mathrm{g}^{(i)}$ are zero. (The models with non Ricci-flat factor spaces were studied by different methods in the papers [92], [99], [100], [88], [101], [86], [102], [95], [103], [104], [105], [97].) Under this assumption the Ricci tensor for the metric (7.118) has following non-zero components [89]

$$R_0^0 = \mathrm{e}^{-2\gamma}\left(\sum_{i=1}^{n} d_i(\dot{\beta}^i)^2 + \ddot{\gamma}_0 - \dot{\gamma}\dot{\gamma}_0\right) \tag{7.119}$$

$$R_{n_i}^{m_i} = \mathrm{e}^{-2\gamma}\left[\ddot{\beta}^i + \dot{\beta}^i(\dot{\gamma}_0 - \dot{\gamma})\right]\delta_{n_i}^{m_i} \tag{7.120}$$

with the definition

$$\gamma_0 = \sum_{i=1}^{n} d_i\beta^i. \tag{7.121}$$

Let $D := 1 + \sum_{i=1}^{n} d_i = \dim M$ denote the total dimension. Then, for $i = 1, \ldots, n$, indices m_i and n_i in (6.9), (6.12) run from $(D - \sum_{j=i}^{n} d_j)$ to $(D - \sum_{j=i}^{n} d_j + d_i)$.

It is well known that the isotropic cosmological model at present time gives a good description of the observables part of the universe. On the other hand, this very fact of our universe's isotropy and homogeneity is puzzling [106]. Even in the papers which are devoted to the problem of inflation, they start mainly with the metric of the isotropic Friedmann universe [107]. However, it is possible that at early stages of its evolution the universe exhibits an anisotropic behaviour [108]. As it was shown in [109, 110], anisotropic cosmological models describe the most general approach to the cosmological singularity (the initial singularity at some instant t_0). Among anisotropic homogeneous models the Kasner solution [111] represents one of the most simple vacuum solutions of the Einstein equations. The Kasner solution is defined on a manifold

$$M = \mathbb{R} \times M_1 \times M_2 \times M_3, \tag{7.122}$$

where the differentiable manifold M_i $(i = 1, 2, 3)$ is either $\mathbb{R}$ or S^1.

Another very puzzling problem is the fact that the space-time of our universe is 4-dimensional. Fashionable theories of unified physical interactions (supergravity or superstrings [112, 113, 114]) use the Kaluza-Klein idea [115, 116] of hidden (or extra) dimensions, according to which our universe at small (Planckian) scales has a dimension more than four. If the extra dimensions are more than a mathematical construct, we should explain what dynamical processes lead from a stage with all dimensions developing with the same scale to the actual stage of the universe, where we have only four external dimensions and all internal spaces have to be compactified and contracted to unobservable scales.

The general structure (7.117) combines both ideas, anisotropy and multidimensionality. There M_i $(i = 1, \ldots, n)$ are d_i-dimensional space of constant curvature (or, more generally, an Einstein space). If $n = 3$ and $d_1 = d_2 = d_3 = 1$ or $n = 2$ and $d_1 = 2$, $d_2 = 1$ then this manifold describes an usual anisotropic homogeneous 4-dimensional space-time. For $n \geq 2$ and a total dimension $D = 1 + \sum_{i=1}^{n} d_i > 4$ we have an anisotropic multidimensional space-time where one of the spaces M_i (say M_1) describes our 3-dimensional external space.

Multidimensional cosmological models of the type (7.117) (with arbitrary n) were investigated intensively in the recent decade (according to our knowledge, starting from the paper [117] investigating the stability of the internal spaces).

Quantization of a multidimensional model with a space-time (7.117) was first performed in [118]. Beside vacuum models, there were also cosmological models considered which contain different types of matter, and exact solutions of the Einstein equations, and of the Wheeler-De Witt equations in the quantum case, were obtained (see [119, 120] and the extended list of references there). Exact solutions are of special interest because they can be used for a detailed study of evolution of the universe (for example in the approach to the singularity), of the compactification of the internal spaces, and of the behaviour of matter fields.

In the present paper we consider an anisotropic homogeneous universe of type (7.117), where all M_i are Ricci-flat. If $n = 3$ and $d_1 = d_2 = d_3 = 1$ it describes the usual 4-dimensional Bianchi type I model. We investigate this space-time in the presence of m non-interacting minimally coupled scalar fields. Scalar fields are now popular in cosmology, because in most inflationary models the presence of a scalar field provides homogeneity, isotropy, and almost spatial flatness of the universe [107]. It was shown in the paper [121] that for a special form of the scalar field potentials these scalar fields are equivalent to a m-component perfect fluid. We exploit this equivalence in [121] to investigate a two-component model (a model with 2 scalar fields). Now we shall integrate this model in the presence of 3 scalar fields where one of them is equivalent to an ultra-stiff perfect fluid, the second one corresponds to dust, and the third one is equivalent to vacuum. The main features of the solutions are the following: If the parameters of the model permit the universe to run from the singularity to infinity, then the universe has a Kasner-like behaviour near the singularity, with isotropization when it goes to infinity. In the 3-component integrable case, the universe has de Sitter-like behaviour in the infinite volume limit. Superficially, it seems this model is not a good candidate for a realistic multidimensional cosmology, because of the isotropization of all directions at late times. But we shall show that there are particular solutions, which describe a birth of the universe from "nothing". The parameters of the model in this case can be chosen in such a way that a scale factor of the external space undergoes inflation, while the other scale factors remain compactified near Planck length. However this model is really only good, if in addition we provide a graceful exit mechanism [122]. For some of the parameters the infinite volume limit takes place in the Euclidean region which has asymptotically anti-de Sitter wormhole geometry. Another interesting Euclidean solution represents an instanton which describes tunnelling between a Kasner-like universe (a baby universe) and an asymptotically de Sitter universe. Sewing a number of these instantons may provide the Coleman mechanism [123] for the vanishing cosmological constant.

The previous paper [124] has already considered multidimensional cosmological models in the presence of a m-component perfect fluid. In the case with one non-Ricci-flat space, say M_1, for $n = 2$ and $d_1 = 2$, $d_2 = 1$, this model describes a usual 4-dimensional Kantowski-Sachs universe (if M_1 has positive constant curvature) or a Bianchi III universe (if M_1 has negative constant curvature). We also found a 3-component integrable model, where the universe has a Kasner-like behaviour near the singularity as in the present paper, but there is no isotropization at all. All scale factors corresponding to Ricci-flat factor spaces M_i are frozen in the infinite volume limit, but the negative curvature space M_1 grows in time. From this point of view, the model does not describe usual 4-dimensional space-time, because of the missing isotropization, but it may be a good candidate for a multidimensional cosmology, if all frozen internal scale factors are near Planck scale. For a positive curvature space M_1, the infinite volume limit takes place in the Euclidean region, which there, in contrast to the present paper, has wormhole geometry only w.r.t. the space M_1, and the wormhole is asymptotically flat.

In the present paper we consider homogeneous minimally coupled scalar fields as a matter source. Usually, real scalar fields are taken. But, it is possible that a purely imaginary scalar field exists too. This implies scalar fields with a negative sign at the kinetic term in the Lagrangian. Such scalar fields may arise after conformal transformation of real scalar fields with arbitrary coupling to gravity [38], [39], [125], [36]. They appear also in the Brans-Dicke theories after the dimensional reduction from higher dimensional theories [114, 126, 26]. Also the C-field of Hoyle and Narlikar has a negative sign in front of the kinetic term [127]. The authors of [128, 129] emphasize the need for scalar fields with negative kinetic terms in multidimensional theories in order to fit the observable data (see also a discussion of this topic in [130]). As we will show here, in the particular case of constant φ, the imaginary scalar field is equivalent to a negative cosmological constant which results in an anti-de Sitter universe. In what follows we do not exclude the possible existence of imaginary scalar fields, whence in our paper we consider real as well as imaginary scalar fields.

We proceed as follows. In Sect. 7.5.1 we describe our model and get an effective perfect fluid Lagrangian, exploiting the equivalence between an m-component perfect fluid and m non-interacting scalar fields with a special class of potentials. In Sect. 7.5.2 we investigate the general dynamics of the universe and its asymptotic behaviour. In Sect. 7.5.3 classical solutions for the integrable 3-component models are obtained. Classical wormhole solutions are obtained in Sect. 7.5.4 where it is also shown that they are asymptotically anti-de Sitter wormholes. Section 7.5.5 is devoted to the reconstruction of the scalar field potentials. Solutions to the quantized models are presented in Sect. 7.5.6.

7.5.1 Multi-component perfect fluid cosmology

Let us consider a cosmological model with a multidimensional metric denoted as

$$g = g_{MN}dx^M \otimes dx^N = -e^{2\gamma(\tau)}d\tau \otimes d\tau + \sum_{i=1}^{n} e^{2\beta^i(\tau)}g^{(i)}, \tag{7.123}$$

where, for $i = 1, \ldots, n$, $g^{(i)} = g^{(i)}_{m_i n_i} dx^{m_i} \otimes dx^{n_i}$, $m_i, n_i = 1, \ldots, d_i$, is the metric form of the Ricci-flat factor space M_i of dimension d_i.

$$R_{m_i n_i}\left[g^{(i)}\right] = 0, \quad i = 1, \ldots n. \tag{7.124}$$

The action of the model be

$$S = \frac{1}{2\kappa^2}\int d^D x\sqrt{|g|}R[g] + S_\varphi + S_{\text{GH}}, \tag{7.125}$$

where S_{GH} is the standard Gibbons-Hawking boundary term, κ^2 is the gravitational coupling constant in dimension $D = \sum_{i=1}^n d_i + 1$, and $S_\varphi = \sum_{a=1}^m S_\varphi^{(a)}$ is the action of m non-interacting minimally coupled homogeneous scalar fields

$$S_\varphi^{(a)} = -\int d^D x\sqrt{|g|}\left[g^{MN}\partial_M\varphi^{(a)}\partial_N\varphi^{(a)} + U^{(a)}(\varphi^{(a)})\right]. \tag{7.126}$$

For the metric (7.123) the action (7.125) reads

$$S = \mu\int d\tau L_s, \tag{7.127}$$

with the Lagrangian

$$L_s = \frac{1}{2}e^{-\gamma+\gamma_0}\left(G_{ij}\dot\beta^i\dot\beta^j + \kappa^2\sum_{a=1}^{m}(\dot\varphi^{(a)})^2\right) - \kappa^2 e^{\gamma+\gamma_0}\sum_{a=1}^{m}U^{(a)}(\varphi^{(a)}). \tag{7.128}$$

Here $\gamma_0 = \sum_{i=1}^n d_i\beta^i$ and $\mu = \prod_{i=1}^n V_i/\kappa^2$ where V_i is the volume of the finite Ricci-flat spaces $(M_i, g^{(i)})$. The components of the minisuperspace metric read

$$G_{ij} = d_i\delta_{ij} - d_i d_j. \tag{7.129}$$

As in [121] we subject the scalar fields to the perfect fluid energy-momentum constraints

$$P^{(a)} = \left(\alpha^{(a)} - 1\right)\rho^{(a)}, \tag{7.130}$$

with constants $\alpha^{(a)}$, $a = 1, \ldots, m$, and the energy densities

$$\rho^{(a)} \equiv -T^{(a)}{}^0_0 = \frac{1}{2}e^{-2\gamma}(\dot\varphi^{(a)})^2 + U^{(a)}(\varphi^{(a)}) \tag{7.131}$$

and momenta

$$P^{(a)} \equiv T^{(a)}{}^M_M = \frac{1}{2}e^{-2\gamma}(\dot{\varphi}^{(a)})^2 - U^{(a)}(\varphi^{(a)}), \quad M = 1, \dots, D-1 \ , \tag{7.132}$$

according to the Lagrangian (7.128). In [121] it was proved that, for cosmological models with a metric (7.123), the presence of m non-interacting scalar fields satisfying the relations (7.130) is equivalent to the presence of an m-component perfect fluid with a Lagrangian

$$L_\rho = \frac{1}{2}e^{-\gamma+\gamma_0}G_{ij}\dot{\beta}^i\dot{\beta}^j - \kappa^2 e^{\gamma+\gamma_0}\sum_{a=1}^{m}\rho^{(a)}, \tag{7.133}$$

and energy densities of the form

$$\rho^{(a)} = A^{(a)}V^{-\alpha^{(a)}}, \qquad a = 1, \dots, m, \tag{7.134}$$

with constants $A^{(a)}$ and a spatial volume scale

$$V = e^{\gamma_0} = \prod_{i=1}^{n} a_i^{d_i} \tag{7.135}$$

defined by the scale factors $a_i = e^{\beta^i}$, $i = 1, \dots, n$. Note, that the total spatial volume is $V_{tot} = \mu \cdot V$. The energy density $\rho^{(a)}$ is then connected with the pressure $P^{(a)}$ via (7.130), and equations (7.131) and (7.132) imply $\alpha^{(a)}\rho^{(a)} = e^{-2\gamma}(\dot{\varphi}^{(a)})^2$. So, for real scalar fields and positive $\alpha^{(a)}$, the energy density of the perfect fluid is positive. But, keeping in mind the possibility of imaginary scalar fields (see Introduction), for the general model we shall also consider the case $\rho^{(a)} < 0$. Then, the constants $A^{(a)}$ may have any sign.

Assuming the speed of sound in each component of the perfect fluid to be less than the speed of light,

$$-|\rho^{(a)}| \ \leq \ P^{(a)} \ \leq \ |\rho^{(a)}|, \qquad a = 1, \dots, m. \tag{7.136}$$

With (7.130) this implies the inequalities

$$0 \ \leq \ \alpha^{(a)} \ \leq \ 2, \qquad a = 1, \dots, m. \tag{7.137}$$

Note that, with $\rho = \sum_{a=1}^{m}\rho^{(a)}$ and $P = \sum_{a=1}^{m}P^{(a)}$, the energy dominance condition requires only $-|\rho| \ \leq \ P \ \leq \ |\rho|$, rather than (7.136). In this paper however, although it might be possible to generalize results for arbitrary $\alpha^{(a)}$, for simplicity we keep the assumption (7.136) in order to make use of the inequalities (7.137).

Exploiting the mentioned equivalence between scalar fields and perfect fluid, we investigate the dynamics of the universe via the Euler-Lagrange equations of (7.133), and reconstruct the scalar field potentials $U^{(a)}(\varphi^{(a)})$ satisfying the perfect fluid constraint (7.130).

7.5.2 General multidimensional dynamics

In the harmonic time gauge $\gamma = \gamma_0 = \sum_{i=1}^n d_i\beta^i$ (see e.g. [118], [38]), the Lagrangian (7.133) with energy densities (7.134) just reads

$$L_\rho = \frac{1}{2}G_{ij}\dot\beta^i\dot\beta^j - \kappa^2 e^{2\gamma_0}\sum_{a=1}^m \rho^{(a)}. \tag{7.138}$$

Then the corresponding scalar (zero energy) constraint can be imposed as

$$\frac{1}{2}G_{ij}\dot\beta^i\dot\beta^j + \kappa^2 e^{2\gamma_0}\sum_{a=1}^m \rho^{(a)} = 0. \tag{7.139}$$

The minisuperspace metric may be diagonalized (see also [118]) to

$$G = \eta_{kl}dz^k \otimes dz^l = -dz^0 \otimes dz^0 + \sum_{i=1}^{n-1} dz^i \otimes dz^i, \tag{7.140}$$

where

$$z^0 = q^{-1}\sum_{j=1}^n d_j\beta^j = q^{-1}\ln V, \tag{7.141}$$

$$z^i = [d_i/\,\Sigma_i\Sigma_{i+1}]^{1/2}\sum_{j=i+1}^n d_j\left(\beta^j - \beta^i\right)\ , \tag{7.142}$$

$i = 1, \dots, n-1$, and

$$q := [(D-1)/(D-2)]^{1/2}, \quad \Sigma_k := \sum_{i=k}^n d_i\ . \tag{7.143}$$

With the aid of these transformations the Lagrangian (7.138) and the scalar constraint (7.139) can be rewritten as

$$L_\rho = \frac{1}{2}\eta_{kl}\dot z^k\dot z^l - \kappa^2\sum_{a=1}^m A^{(a)}\exp(k^{(a)}qz^0), \tag{7.144}$$

$$\frac{1}{2}\eta_{kl}\dot z^k\dot z^l + \kappa^2\sum_{a=1}^m A^{(a)}\exp(k^{(a)}qz^0) = 0 \tag{7.145}$$

respectively. Here, $k^{(a)} := 2-\alpha^{(a)}$ $(a = 1, \dots, m)$, whence the inequalities (7.137) for $\alpha^{(a)}$ hold also for $k^{(a)}$,

$$0 \ \leq\ k^{(a)}\ \leq\ 2. \tag{7.146}$$

The equations of motion for z^i, $i = 1, \dots, n-1$, simply read

$$\ddot z^i = 0, \tag{7.147}$$

and readily yield

$$z^i = p^i\tau + q^i, \tag{7.148}$$

where τ is the harmonic time, p^i and q^i are constants. Clearly the geometry is real if p^i and q^i are real. The dynamics of z^0 is then given by the scalar constraint (7.145), which may now be written as

$$-\frac{1}{2}(\dot{z}^0)^2 + \varepsilon + \kappa^2 \sum_{a=1}^{m} A^{(a)} \exp(k^{(a)} q z^0) = 0, \tag{7.149}$$

for a real geometry with

$$\varepsilon := \frac{1}{2}\sum_{i=1}^{n-1}(p^i)^2 \geq 0. \tag{7.150}$$

The coordinate transformations (7.141) and (7.142) can be written as

$$z^k = \sum_{i=1}^{n} t^k{}_i \beta^i \ , \quad k = 0, \ldots, n-1, \tag{7.151}$$

whence the inverse is given by

$$\beta^i = \sum_{k=0}^{n-1} \bar{t}^i{}_k z^k \ , \quad i = 1, \ldots, n. \tag{7.152}$$

For $i = 1, \ldots, n$, with $t^0{}_i = d_i/q$ and $\bar{t}^i{}_0 = [q(D-2)]^{-1}$ we obtain the scale factors

$$a_i = A_i V^{1/(D-1)} e^{\alpha^i \tau}, \tag{7.153}$$

where

$$A_i := e^{\gamma^i}, \quad \gamma^i := \sum_{l=1}^{n-1} \bar{t}^i{}_l q^l, \quad \alpha^i := \sum_{l=1}^{n-1} \bar{t}^i{}_l p^l. \tag{7.154}$$

The parameters α^i satisfy the relations

$$\sum_{i=1}^{n} d_i \alpha^i = 0, \tag{7.155}$$

$$\sum_{i=1}^{n} d_i (\alpha^i)^2 = \sum_{l=1}^{n-1} (p^l)^2 = 2\varepsilon \ , \tag{7.156}$$

and, analogously the parameters γ^i fulfil

$$\sum_{i=1}^{n} d_i \gamma^i = 0, \tag{7.157}$$

$$\sum_{i=1}^{n} d_i (\gamma^i)^2 = \sum_{l=1}^{n-1} (q^l)^2. \tag{7.158}$$

From the definition (7.154) and the relation (7.157) it follows that

$$\prod_{i=1}^{n} A_i^{d_i} = 1. \tag{7.159}$$

Note that using the constraints (7.155) and (7.159) the equation (7.153) yields again (7.135), i.e. $\prod_{i=1}^{n} a_i^{d^i} = V$. Recall that τ in (7.153) is the harmonic time. The synchronous and harmonic times are related by

$$t = \pm \int e^{\gamma_0} d\tau + t_0 = \pm \int V d\tau + t_0. \qquad (7.160)$$

The expression (7.153) shows that the general model does not belong to a class with static internal spaces (see e.g. [131]), but just for $\varepsilon = 0$ (i.e. $\alpha^i = 0$, $i = 1, \ldots, n$), there is a solution

$$a_i = A_i V^{1/(D-1)}, \quad i = 1, \ldots, n, \qquad (7.161)$$

which is isotropic.

In order to find the dynamical behaviour of the universe we should now solve the constraint (7.149), i.e. the mechanical energy conservation equation

$$\varepsilon = T + U \qquad (7.162)$$

with kinetic energy $T := \frac{1}{2}(\dot{z}^0)^2$ and potential $U := -\kappa^2 \sum_{a=1}^{m} A^{(a)} \exp(k^{(a)} q z^0)$. Depending on the parameters $A^{(a)}$ and their signs, the potential U may exhibit a rich structure with several extrema, and a classical Lorentzian trajectory is bound by possible turning points at $\varepsilon = U$. Since the general dynamics is very complex, we investigate the asymptotic behaviour of our model universe in the limit of large spatial geometries $V \to \infty$ and near the singularity $V \to 0$. Without restriction we suppose now

$$0 \le k^{(1)} < \ldots < k^{(m)} \le 2. \qquad (7.163)$$

<u>1. Limit $V \to \infty$:</u> In the limit $V \to \infty$ (i.e. $z_0 \to \infty$) the term $-\kappa^2 A^{(m)} \exp(k^{(m)} q z^0)$ dominates the potential U, whence, for $k^{(m)} \neq 0$, there are two cases to be distinguished:

<u>i) $A^{(m)} > 0$:</u> Here, for $V \to \infty$, the term ε may be neglected. So, the constraint equation (7.162) has the asymptotic solution

$$e^{q z_0} = V \approx (2\kappa^2 A^{(m)})^{-1/k^{(m)}} (\bar{q}|\tau|)^{-2/k^{(m)}}, \qquad (7.164)$$

with $2\bar{q} := k^{(m)} q$, where (without restriction) we have chosen initial conditions such that $V \to \infty$ at $\tau \to 0$, when according to (7.153) the system is subject to an isotropization,

$$a_i \sim V^{1/(D-1)}, \quad V \to \infty, \quad i = 1, \ldots, n\ . \qquad (7.165)$$

In this limit the harmonic and synchronous times are connected by

$$|\tau| \sim |t|^{k^{(m)}/(k^{(m)}-2)}\ , \quad k^{(m)} \neq 2\ , \qquad (7.166)$$

$$|\tau| \sim \exp\left(-\sqrt{2\kappa^2 A^{(m)}} q |t|\right)\ , \quad k^{(m)} = 2\ . \qquad (7.167)$$

So, the synchronous time evolution of the spatial volume is (asymptotically for $t \to \infty$) given by

$$V \sim |t|^{2/\alpha^{(m)}}, \quad k^{(m)} \neq 2\ , \qquad (7.168)$$

$$V \sim \exp\left(\sqrt{2\kappa^2 A^{(m)}} q |t|\right), \quad k^{(m)} = 2\ , \qquad (7.169)$$

with scale factors (according to isotropization)

$$a_i \sim |t|^{2/\alpha^{(m)}(D-1)} , \quad k^{(m)} \neq 2 , \tag{7.170}$$

$$a_i \sim \exp\big(\frac{\sqrt{2\kappa^2 A^{(m)}}q}{D-1}|t|\big) , \quad k^{(m)} = 2 . \tag{7.171}$$

Taking a usual anisotropic space-time model ($D = 4$, $n = 3$, $d_1 = d_2 = d_3 = 1$) then for large (synchronous) times the formulas (7.170) and (7.171) yield scale factors $a_i \sim |t|^{2/3}$ for $k^{(m)} = 1$ (dust) and $a_i \sim \exp\big(\sqrt{\kappa^2 A^{(m)}/3}|t|\big)$ for $k^{(m)} = 2$ (vacuum). Asymptotically, power-law inflation (with power $p > 1$) takes place for $0 < \alpha^{(m)} < 2/(D-1)$, and $\alpha^{(m)} = 2/(D-1)$ yields a generalized Milne universe.

ii) $A^{(m)} < 0$: Here, the Lorentzian region has a boundary at the turning point $V_{\max}$ of the volume scale, which in the large energy limit $\varepsilon \to \infty$ is asymptotically given as

$$V_{\max} \approx \left[\frac{\varepsilon}{\kappa^2 |A^{(m)}|}\right]^{1/k^{(m)}} . \tag{7.172}$$

The region with $V > V_{\max}$ is the Euclidean sector. For $V >> V_{\max}$, we obtain the asymptotically isotropic solution

$$a_i \sim V^{1/(D-1)} \approx \left[\sqrt{2\kappa^2 |A^{(m)}|}\bar{q}|\tau|\right]^{-2/k^{(m)}(D-1)} . \tag{7.173}$$

In the Euclidean region, we obtain a classical wormhole w.r.t. each factor space. With constants of integration (in (7.148)) $p_i = 0$ (i.e. $\alpha^i = 0$), $i = 1, \ldots, n$, the wormhole takes its most simple and symmetric form. Then the throats are given by

$$a_{(\mathrm{th})i} \approx A_i \left[\varepsilon/\kappa^2 |A^{(m)}|\right]^{1/k^{(m)}(D-1)} . \tag{7.174}$$

In the case $k^{(m)} = 2$ we obtain asymptotically (for $t \to \infty$) anti-de Sitter wormholes with synchronous time scale factors

$$a_i \sim \exp\big(\frac{\sqrt{2\kappa^2 |A^{(m)}|}q}{D-1}|t|\big) , \quad i = 1, \ldots, n . \tag{7.175}$$

2. Limit $V \to 0$: For $k^{(1)} \neq 0$, in the small volume limit $V \to 0$, i.e. $z^0 \to -\infty$, the potential vanishes $U \to 0$. So, for $\varepsilon > 0$, we obtain (asymptotically for $t \to 0$) a (multi-dimensional) Kasner universe [83, 89], with scale factors

$$a_i \sim |t|^{\bar{\alpha}^i} , \quad i = 1, \ldots, n . \tag{7.176}$$

with parameters $\bar{\alpha^i}$ satisfying

$$\sum_{i=1}^{n} d_i \bar{\alpha^i} = 1 , \quad \sum_{i=1}^{n} d_i (\bar{\alpha^i})^2 = 1 . \tag{7.177}$$

If $k^{(1)} = 0$, then $U \to -\kappa^2 A^{(1)}$ for $z^0 \to -\infty$. Here, for $E := \varepsilon + \kappa^2 A^{(1)} > 0$, we obtain (asymptotically for $t \to 0$) a generalized Kasner universe [83], i.e. scale factors (7.176) with parameters $\bar{\alpha}^i$ satisfying

$$\sum_{i=1}^{n} d_i \bar{\alpha}^i = 1 \ , \quad \sum_{i=1}^{n} d_i (\bar{\alpha}^i)^2 = 1 - \bar{\alpha}^2, \tag{7.178}$$

with the parameter $\bar{\alpha} \to 0$ for $A^{(1)} \to 0$.

In the exceptional case $E = \varepsilon + \kappa^2 A^{(1)} = 0$ the term of the matter component $a = 2$ dominates the constraint (7.149), whence we obtain (compare also [121])

$$V \sim t^{2/\alpha^{(2)}} \ , \tag{7.179}$$

$$a_i \sim t^{2/[(D-1)\alpha^{(2)}]} \exp\left\{ \alpha^i f(\alpha^{(2)}) t^{-\frac{2-\alpha^{(2)}}{\alpha^{(2)}}} \right\}, \quad i = 1, \ldots, n \ , \tag{7.180}$$

where $f(x) := \left(\frac{2-x}{x}\right)^{\frac{(2-x)}{x}} \left[\frac{2}{(2-x)^2 q^2 \kappa^2 A^{(2)}}\right]^{\frac{1}{x}}$. In another exceptional case where $\varepsilon = 0$ (i.e. $\alpha^i = 0$, $i = 1, \ldots, n$) the universe is isotropic everywhere, i.e. $a_i \sim V^{1/D-1}$, $i = 1, \ldots, n$. If, for example, $k^{(1)} = 0$ (and $A^{(1)} = 0$) we obtain from (7.179) or (7.180)

$$a_i \sim t^{2/[(D-1)\alpha^{(2)}]} \ . \tag{7.181}$$

7.5.3 Integrable 3-component model: Classical solutions

In this section, we consider the integrable case of a three-component perfect fluid ($m = 3$) where one of them ($a = 1$) is ultra-stiff matter ($k^{(1)} = 0, \alpha^{(1)} = 2$), the second one ($a = 2$) is dust ($k^{(2)} = 1, \alpha^{(2)} = 1$), and the third one ($a = 3$) is vacuum ($k^{(3)} = 2, \alpha^{(3)} = 0$). The case $k^{(1)} = 0, k^{(3)} = 2k^{(2)}$ with $0 < k^{(2)} \leq 2$ is also included if one substitutes q by $\bar{q} = k^{(2)} q$.

The constraint equation (7.149) reads in this case

$$-\frac{1}{2} \left(\dot{z}^0\right)^2 + \varepsilon + \kappa^2 A^{(1)} + \kappa^2 A^{(2)} e^{q z^0} + \kappa^2 A^{(3)} e^{2 q z^0} = 0 \tag{7.182}$$

and can be rewritten like

$$E = \frac{1}{2} \left(\dot{z}^0\right)^2 + U(z^0) \,, \tag{7.183}$$

where

$$E := \varepsilon + \kappa^2 A^{(1)} \tag{7.184}$$

and the potential $U(z^0)$ is

$$U(z^0) := -B e^{q z^0} - C e^{2 q z^0} \tag{7.185}$$

with the definitions $B := \kappa^2 A^{(2)}$ and $C := \kappa^2 A^{(3)}$.

As mentioned in the introduction, for a complete description of the model the parameters E, B, and C are considered to take positive and negative values. Then, we have four qualitatively different shapes of the potential (7.185) (see Fig. 7 and Fig. 8). For each of them, we shall solve the constraint equation separately. Eq. (7.183) integrates to

$$\int \frac{dV}{V\sqrt{E+BV+CV^2}} = \pm\sqrt{2}q(\tau-\tau_0)\,, \tag{7.186}$$

where τ is the harmonic time coordinate, and τ_0 is a constant of integration.

<u>i) $B>0$, $C>0$ (see Fig. 7):</u> The solutions of equations (7.186) are

$$V = \frac{1}{B}\frac{1}{\frac{q^2}{2}\left[f^2 - \frac{2C}{q^2B^2}\right]}, \qquad E=0, \tag{7.187}$$

$$V = \frac{4Ef}{(B-f)^2-4EC}, \qquad E>0, \tag{7.188}$$

$$V = \frac{2|E|}{B+\sqrt{|\Delta|}f}, \qquad E<0, \tag{7.189}$$

where $\Delta := 4EC - B^2$ ($|\Delta| = B^2+4|E|C$ for $E<0$) and

$$f = \tau-\tau_0, \quad E=0, \quad \frac{\sqrt{2C}}{qB} \le |\tau-\tau_0| < +\infty\,, \tag{7.190}$$

$$f = \exp\left(\sqrt{2E}q(\tau-\tau_0)\right), \quad E>0, \quad \ln\left(B+2\sqrt{EC}\right) \le \ln f < +\infty\,, \tag{7.191}$$

$$f = \sin\left(\sqrt{2|E|}q(\tau-\tau_0)\right), \quad E<0, \quad -\arcsin\left(\frac{B}{\sqrt{|\Delta|}}\right) \le \arcsin f \le \frac{\pi}{2}\,. \tag{7.192}$$

The synchronous and harmonic time coordinate are related via

$$\tau-\tau_0 = \frac{\sqrt{2C}}{qB}\coth\left(\sqrt{\frac{C}{2}}qt\right), \quad E=0, \tag{7.193}$$

$$\exp\left(\sqrt{2E}q(\tau-\tau_0)\right) = B+\sqrt{4EC}\coth\left(\sqrt{\frac{C}{2}}qt\right), \quad E>0, \tag{7.194}$$

$$\tan\left(\sqrt{|E|/2}q(\tau-\tau_0)\right) = \frac{\sqrt{|\Delta|}}{B}\left[\sqrt{\frac{4|E|C}{|\Delta|}}\coth\left(\sqrt{\frac{C}{2}}qt\right)-1\right], \quad E<0. \tag{7.195}$$

Using these relations, we obtain the expressions for the volume of the universe in synchronous time:

$$V = \frac{B}{C}\sinh^2\left(\sqrt{\frac{C}{2}}qt\right), \quad E=0, \quad |t|<\infty, \tag{7.196}$$

$$V = \frac{1}{C}\left[B+\sqrt{4EC}\coth\left(\sqrt{\frac{C}{2}}qt\right)\right]\sinh^2\left(\sqrt{\frac{C}{2}}qt\right), \quad E>0, \quad 0\le t<+\infty, \tag{7.197}$$

$$V = \frac{2|E|(1+\tan^2(y/2))}{B(1+\tan^2(y/2))+2\sqrt{|\Delta|}\tan(y/2)}, \quad E<0, \tag{7.198}$$

where $\tan(y/2) = \tan\left(\sqrt{|E|/2}q(\tau - \tau_0)\right)$ is given by (7.195). Expression (7.198) can be written in a more convenient way if the parameter τ_0 is chosen such that equation (7.189) is symmetric with respect to the turning point $V_0 = \left(-B + \sqrt{|\Delta|}\right)/2C$, namely

$$V = \frac{2|E|}{B + \sqrt{|\Delta|}\cos\left(\sqrt{2|E|}q\tau\right)}, \quad |\tau| < \frac{1}{\sqrt{2|E|}q}\left[\frac{\pi}{2} + \arcsin\left(\frac{B}{\sqrt{|\Delta|}}\right)\right]. \tag{7.199}$$

In this case,

$$\tan\left(\sqrt{|E|/2}q\tau\right) = \frac{\sqrt{4|E|C}}{\sqrt{|\Delta|} - B}\tanh\left(\sqrt{\frac{C}{2}}qt\right) \tag{7.200}$$

and for the volume results

$$V = \frac{1}{2C}\left[\sqrt{|\Delta|} - B + \left(\sqrt{|\Delta|} + B\right)\tanh^2\left(\sqrt{\frac{C}{2}}qt\right)\right]\cosh^2\left(\sqrt{\frac{C}{2}}qt\right), \quad |t| < \infty. \tag{7.201}$$

The region $V < V_0$ is the Euclidean sector and we obtain the instanton by analytic continuation $t \to -it$ in formula (7.201):

$$V = \frac{1}{2C}\left[\sqrt{|\Delta|} - B - \left(\sqrt{|\Delta|} + B\right)\tan^2\left(\sqrt{\frac{C}{2}}qt\right)\right]\cos^2\left(\sqrt{\frac{C}{2}}qt\right) \tag{7.202}$$

with $|t| \leq \frac{2}{\sqrt{2C}q}\arctan\sqrt{\frac{\sqrt{|\Delta|}-B}{\sqrt{|\Delta|}+B}}$.

On the quantum level, this instanton is responsible for the birth of the universe from "nothing".

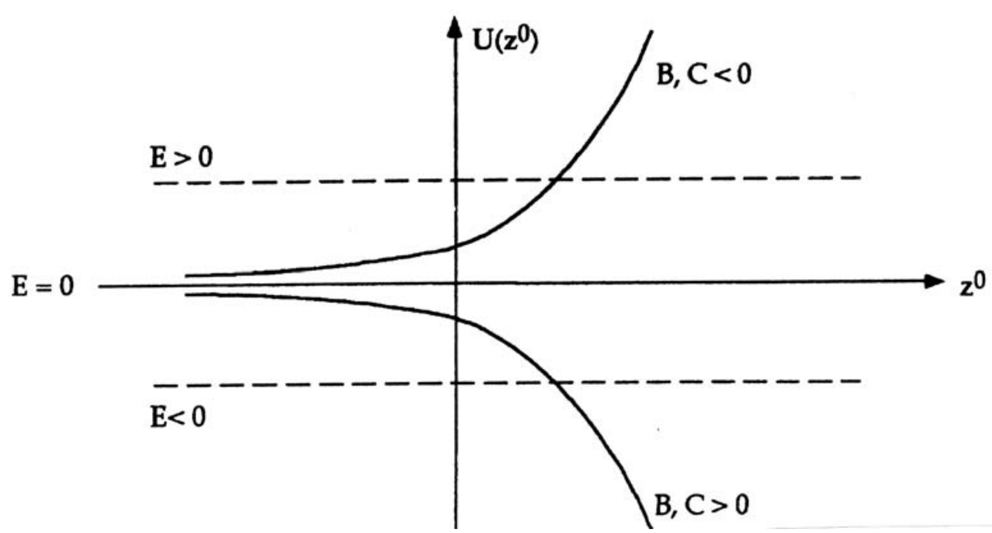

Figure 7: The potential $U_0(z^0)$ (solid line) and the energy levels E (dashed lines) in the cases $B, C > 0$ and $B, C < 0$. Lorentzian regions exist for $E > U_0(z^0)$.

ii) $B < 0$, $C > 0$ (see Fig. 8): In this case, the maximal value of the potential $U(z^0)$ is $U_m = B^2/4C$ at $z_m^0 = \frac{1}{q}\ln(|B|/2C)$ and for $0 < E < V_m$ we have two turning points, namely:

$$V_0^{(1,2)} = \left(|B| \pm \sqrt{|\Delta|}\right)/2C\,, \tag{7.203}$$

where $|\Delta| = B^2 - 4EC$. Classical motion takes place either for $0 \le V \le V_0^{(1)}$ or for $V_0^{(2)} \le V < +\infty$.

If $E \le 0$, we have one turning point only, namely

$$V_0 = |B|/C, \quad E = 0, \tag{7.204}$$

$$V_0 = \left(|B| + \sqrt{|\Delta|}\right)/2C, \quad E < 0\,, \tag{7.205}$$

where $|\Delta| = B^2 + 4|E|C$ and classical motion takes place for $V \ge V_0$.

The solutions of the equation (7.186) read

$$V = \frac{1}{|B|}\frac{1}{\frac{q^2}{2}\left(\frac{2C}{B^2q^2} - f^2\right)}, \qquad E = 0, \tag{7.206}$$

$$V = \frac{4Ef}{(|B|+f)^2 - 4EC}, \qquad 0 < E < U_m, \quad 0 \le V \le V_0^{(1)}, \tag{7.207}$$

$$V = \frac{4Ef}{4EC - (|B|-f)^2}, \qquad 0 < E < U_m, \quad V_0^{(2)} \le V < +\infty, \tag{7.208}$$

$$V = \frac{4Ef}{(|B|+f)^2 - 4EC}, \qquad E > U_m, \tag{7.209}$$

$$V = \frac{2|E|}{\sqrt{|\Delta|}f - |B|}, \qquad E < 0, \tag{7.210}$$

where

$$f = \tau - \tau_0, \qquad E = 0, \quad |\tau - \tau_0| < \sqrt{2C}/q|B|, \tag{7.211}$$

$$f = \exp\left(\sqrt{2E}q(\tau - \tau_0)\right), \qquad 0 < E < U_m, \quad V \le V_0^{(1)}, \quad \ln\sqrt{|\Delta|} \le \ln f < +\infty, \tag{7.212}$$

$$f = \exp\left(\sqrt{2E}q(\tau - \tau_0)\right), \qquad 0 < E < U_m, \quad V \ge V_0^{(2)}, \quad \ln\left(|B| - \sqrt{4EC}\right) \le \ln f \le \ln\sqrt{|\Delta|}, \tag{7.213}$$

$$f = \exp\left(\sqrt{2E}q(\tau - \tau_0)\right), \qquad E > U_m, \quad \ln\left(-|B| + \sqrt{4EC}\right) \le \ln f < +\infty, \tag{7.214}$$

$$f = \sin\left(\sqrt{2|E|}q(\tau - \tau_0)\right), \qquad E < 0, \quad \arcsin\frac{|B|}{\sqrt{|\Delta|}} \le \arcsin f \le \frac{\pi}{2}. \tag{7.215}$$

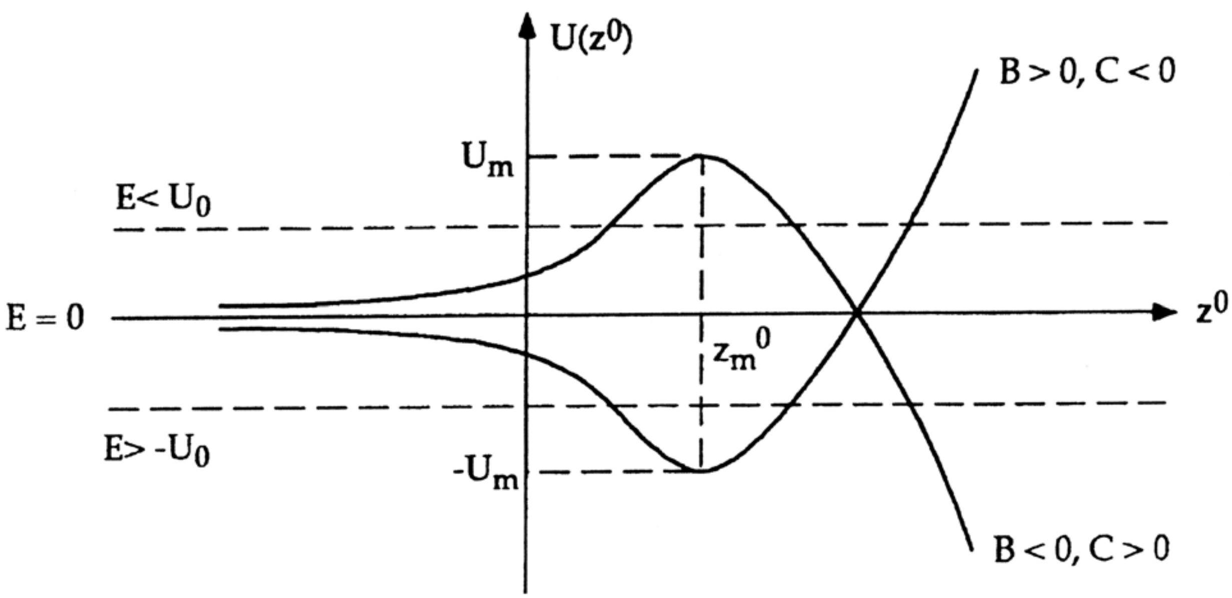

Figure 8: The potential $U_0(z^0)$ (solid line) and the energy levels E (dashed lines) in the cases $B > 0$, $C < 0$ and $B < 0$, $C > 0$. In the former case we get a potential well, and in the latter case we obtain a potential barrier. Lorentzian regions exist for $E > U_0(z^0)$. Here, $U_m = B^2/4C$ and $z^0_m = \frac{1}{q}\ln|B/2C|$.

The harmonic and synchronous time coordinates are related via

$$\tau = \frac{2C}{|B|q}\tanh\left(\sqrt{\frac{C}{2}}qt\right), \qquad E = 0, \tag{7.216}$$

$$f = \sqrt{4EC}\cot\left(\sqrt{\frac{C}{2}}qt\right) - |B|, \qquad 0 < E < B^2/4C, \quad V < V_0^{(1)}, \tag{7.217}$$

$$f = -\sqrt{4EC}\tanh\left(\sqrt{\frac{C}{2}}qt\right) + |B|, \qquad 0 < E < B^2/4C, \quad V > V_0^{(2)}, \tag{7.218}$$

$$f = \sqrt{4EC}\coth\left(\sqrt{\frac{C}{2}}qt\right) - |B|, \qquad E > B^2/4C, \tag{7.219}$$

$$\tan\left(\sqrt{|E|/2}q\tau\right) = \frac{\sqrt{4|E|C}}{\sqrt{|\Delta|} + |B|}\tanh\left(\sqrt{\frac{C}{2}}qt\right), \qquad E < 0, \tag{7.220}$$

where in (7.216) and (7.220) the constant τ_0 is chosen such that the expressions are symmetric with respect to the turning point at the minimum. Then, the volume of the universe is

$$V = \frac{|B|}{C}\cosh^2\left(\sqrt{\frac{C}{2}}qt\right), \quad E = 0, \quad |t| < +\infty, \tag{7.221}$$

$$V = \frac{1}{C}\left[\sqrt{4EC}\coth\left(\sqrt{\frac{C}{2}}qt\right) - |B|\right]\sinh^2\left(\sqrt{\frac{C}{2}}qt\right), \tag{7.222}$$

$$0 < E < B^2/4C, \quad V < V_0^{(1)}, \quad 0 \le t \le \frac{2}{\sqrt{2C}q}\mathrm{arcoth}\frac{|B| + \sqrt{|\Delta|}}{\sqrt{4EC}},$$

$$V = \frac{1}{2C}\left[|B| + \sqrt{|\Delta|} - \left(|B| - \sqrt{|\Delta|}\right)\tanh^2\left(\sqrt{\frac{C}{2}}qt\right)\right]\cosh^2\left(\sqrt{\frac{C}{2}}qt\right), \tag{7.223}$$

$$0 < E < B^2/4C, \quad V > V_0^{(2)}, \quad |t| < +\infty$$

$$V = \frac{1}{C}\left[\sqrt{4EC}\coth\left(\sqrt{\frac{C}{2}}qt\right) - |B|\right]\sinh^2\left(\sqrt{\frac{C}{2}}qt\right), \tag{7.224}$$

$$E > B^2/4C, \quad 0 \le t < +\infty$$

$$V = \frac{1}{2C}\left[\sqrt{|\Delta|} + |B| + \left(\sqrt{|\Delta|} - |B|\right)\tanh^2\left(\sqrt{\frac{C}{2}}qt\right)\right]\cosh^2\left(\sqrt{\frac{C}{2}}qt\right), \tag{7.225}$$

$$E < 0, \quad |t| < +\infty.$$

Eqs. (7.221), (7.223), and (7.225) are written in a symmetric way with respect to the turning point at $t = 0$. The instanton solutions can be obtained by analytic continuation of these symmetric expressions and result in:

$$V = \frac{|B|}{C}\cos^2\left(\sqrt{\frac{C}{2}}qt\right), \quad E = 0, \quad |t| \le \pi/\sqrt{2C}q, \tag{7.226}$$

$$V = \frac{1}{2C}\left[|B| + \sqrt{|\Delta|} + \left(|B| - \sqrt{|\Delta|}\right)\tan^2\left(\sqrt{\frac{C}{2}}qt\right)\right]\cos^2\left(\sqrt{\frac{C}{2}}qt\right), \tag{7.227}$$

$$0 < E < B^2/4C \quad (|\Delta| = B^2 - 4EC), \quad |t| \le \pi/\sqrt{2C}q,$$

$$V = \frac{1}{2C}\left[|B| + \sqrt{|\Delta|} + \left(|B| - \sqrt{|\Delta|}\right)\tan^2\left(\sqrt{\frac{C}{2}}qt\right)\right]\cos^2\left(\sqrt{\frac{C}{2}}qt\right), \tag{7.228}$$

$$E < 0 \quad (|\Delta| = B^2 + 4|E|C), \quad |t| \le \frac{2}{\sqrt{2C}q}\arctan\sqrt{\frac{\sqrt{|\Delta|} + |B|}{\sqrt{|\Delta|} - |B|}}.$$

In equation (7.227), the instanton is symmetric with respect to the turning point $V_0^{(2)}$. For the same instanton but now symmetric with respect to the turning point at $V_0^{(1)}$, we have

$$V = \frac{1}{2C}\left[|B| - \sqrt{|\Delta|} + \left(|B| + \sqrt{|\Delta|}\right)\tan^2\left(\sqrt{\frac{C}{2}}qt\right)\right]\cos^2\left(\sqrt{\frac{C}{2}}qt\right), \tag{7.229}$$

$$0 < E < B^2/4C, \quad |t| \le \pi/\sqrt{2C}q.$$

All the instantons (7.226) to (7.229) are responsible on the quantum level for the birth of the universe from “nothing”. The instanton (7.227), (7.229) is of special interest. Its qualitative shape is seen in Fig. 9 where $V_{min} = V_0^{(1)}$ and $V_{max} = V_0^{(2)}$. The instanton describes tunneling between a multidimensional Kasner-like universe (a baby universe) and a multidimensional de Sitter universe because, as was mentioned in Sect 7.5.2 and as we shall demonstrate more precisely latter, the limit $V \to 0$ corresponds to a Kasner-like universe and in the limit $V \to \infty$ we obtain an (isotropic) de Sitter universe (in [132] to [135] analogous types of an instanton describing tunneling between a Friedmann universe and a de Sitter universe were obtained for a different model). The instanton may also represent the birth (demise) of a de Sitter universe (see (7.227)) and a baby universe (see (7.229)) from (into) "nothing". As was demonstrated in [132, 133], the instanton may be extended beyond $V = V_{min,max}$ gluing together a number of Euclidean manifolds (see Fig. 10). Such gluing may provide the Coleman mechanism [123] that establishes the vanishing of the cosmological constant.

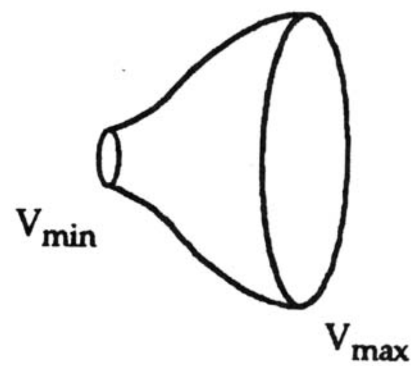

Figure 9: The qualitative shape of the instanton (7.227) (or (7.229)). The instanton describes tunnelling between a Kasner-like (baby) universe and a de Sitter universe.

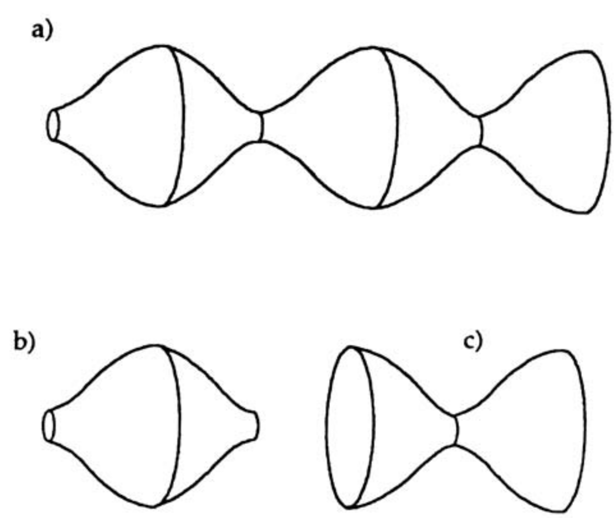

Figure 10: Examples of instantons constructed by sewing together several copies of the the instanton illustrated in Fig. 9. Such instantons may describe tunnelling between (a) Kasner and de Sitter universe, (b) two Kasner universes, (c) two de Sitter universes.

The case with $E = B^2/4C$ is degenerated. Here, two turning points coincide with each other: $V_0^{(1)} = V_0^{(2)} \equiv V_0 = |B|/2C$. In this case, we obtain with the synchronous time coordinate t for the two volumes the expressions

$$V = \frac{2E}{|B|}\left[1 - \exp\left(-\frac{|B|}{\sqrt{2E}}qt\right)\right], \quad 0 \le t < +\infty, \tag{7.230}$$

which describes an infinitely long lasting rolling down from the unstable equilibrium position V_0 to the singularity $V = 0$ and

$$V = \frac{2E}{|B|}\left[1 + \exp\left(\frac{|B|}{\sqrt{2E}}qt\right)\right], \quad |t| < +\infty \tag{7.231}$$

describing the infinitely long lasting rolling down with $V \to \infty$.

To obtain solutions in the two remaining cases $B > 0, \quad C < 0$ (see Fig. 8) and $B < 0, \quad C < 0$ (see Fig. 7), it is not necessary to solve equation (7.186) again. We can instead take the solutions found already in subsections i) and ii). It is clear that the Euclidean solutions obtained there are Lorenzian ones here and vice versa Lorenzian solutions of subsections i) and ii) are Euclidean ones here. What we have to do is the evident substitutions $B \to |B|$, $C \to |C|$, and $E \to -E$ where it is necessary. For example:

iii) $B > 0$, $C < 0$: From (7.226), (7.228), and (7.229), we obtain respectively

$$V = \frac{B}{|C|}\cos^2\left(\sqrt{|C|/2}qt\right), \quad E = 0, \quad |t| \le \pi/q\sqrt{2|C|}, \tag{7.232}$$

$$V = \frac{1}{2|C|}\left[\sqrt{|\Delta|} + B - \left(\sqrt{|\Delta|} - B\right)\tan^2\left(\sqrt{|C|/2}qt\right)\right]\cos^2\left(\sqrt{|C|/2}qt\right), \tag{7.233}$$

$$E > 0, \quad |t| \le \frac{2}{\sqrt{2|C|}q}\arctan\left[\left(\sqrt{|\Delta|} + B\right)/\left(\sqrt{|\Delta|} - B\right)\right]^{1/2},$$

$$V = \frac{1}{2|C|}\left[B - \sqrt{|\Delta|} + \left(B + \sqrt{|\Delta|}\right)\tan^2\left(\sqrt{|C|/2}qt\right)\right]\cos^2\left(\sqrt{|C|/2}qt\right), \tag{7.234}$$

$$-B^2/4|C| < E < 0, \quad |t| < \pi/\sqrt{2|C|}q\,.$$

Solution (7.233) is symmetric with respect to the classical turning point. To investigate the limit $|t| \to 0$ it is better to give another representation of the same solution, namely

$$V = \frac{1}{|C|}\left[B + \sqrt{4E|C|}\cot\left(\sqrt{|C|/2}qt\right)\right]\sin^2\left(\sqrt{|C|/2}qt\right), \tag{7.235}$$

$$E > 0, \quad 0 \le t \le \frac{2}{\sqrt{2|C|}q}\arctan\frac{\sqrt{|\Delta|} - B}{\sqrt{4E|C|}}.$$

In this case, the harmonic time coordinate and the synchronous one are related via

$$\exp\left(\sqrt{2E}q(\tau - \tau_0)\right) = B + \sqrt{4E|C|}\cot\left(\sqrt{|C|/2}qt\right). \tag{7.236}$$

Equation (7.234) is symmetric with respect to the turning point $V_0^{(1)} = \left(B - \sqrt{|\Delta|}\right)/2|C|$. Its analytic continuation gives a "parent instanton" (see (7.222)) with

$$
\begin{aligned}
V &= \frac{1}{|C|}\left[\sqrt{4EC}\coth\left(\sqrt{|C|/2}qt\right) - B\right]\sinh^2\left(\sqrt{|C|/2}qt\right), \qquad (7.237)\\
& -B^2/4|C| < E < 0, \quad 0 \le t \le \frac{2}{\sqrt{2|C|}q}\operatorname{arcoth}\frac{B+\sqrt{|\Delta|}}{\sqrt{4EC}},
\end{aligned}
$$

which is responsible for the birth of a baby universe from "nothing". The Lorenzian solution (7.234) symmetrically written with respect to the turning point $V_0^{(2)} = \left(B + \sqrt{|\Delta|}\right)/2|C|$ reads (see (7.227))

$$
\begin{aligned}
V &= \frac{1}{2|C|}\left[B + \sqrt{|\Delta|} + \left(B - \sqrt{|\Delta|}\right)\tan^2\left(\sqrt{|C|/2}qt\right)\right]\cos^2\left(\sqrt{|C|/2}qt\right), \qquad (7.238)\\
& -B^2/4|C| < E < 0, \quad |t| \le \pi/\sqrt{2|C|}q.
\end{aligned}
$$

<u>iv) $B < 0$, $C < 0$:</u> Here, a Lorenzian region exists for $E > 0$ only. From equation (7.186), we obtain

$$
\begin{aligned}
V &= \frac{1}{|C|}\left[4E|C|\cot\left(\sqrt{|C|/2}qt\right) - |B|\right]\sin^2\left(\sqrt{|C|/2}qt\right), \qquad (7.239)\\
& E > 0, \quad 0 \le t \le \frac{2}{\sqrt{2|C|}q}\operatorname{arccot}\frac{\sqrt{|\Delta|}+|B|}{4E|C|}
\end{aligned}
$$

and the equation relating the harmonic time coordinate and the synchronous one reads

$$
\exp\left(\sqrt{2E}q(\tau-\tau_0)\right) = \sqrt{4E|C|}\cot\left(\sqrt{|C|/2}qt\right) - |B|. \qquad (7.240)
$$

These equations are useful for the investigation of the small time limit $|t| \to 0$.

To obtain an instanton solution (wormhole) it is necessary to rewrite equation (7.239) symmetrically with respect to the classical turning point $V_0 = \left(-|B| + \sqrt{|\Delta|}\right)/2|C|$. We can reformulate equation (7.239) or use directly equation (7.202). The result is

$$
\begin{aligned}
V &= \frac{1}{2|C|}\left[\sqrt{|\Delta|} - |B| - \left(\sqrt{|\Delta|}+|B|\right)\tan^2\left(\sqrt{|C|/2}qt\right)\right]\cos^2\left(\sqrt{|C|/2}qt\right), \qquad (7.241)\\
& E > 0, \quad |t| \le \frac{2}{\sqrt{2|C|}q}\arctan\sqrt{\frac{\sqrt{|\Delta|}-|B|}{\sqrt{|\Delta|}+|B|}}.
\end{aligned}
$$

We shall investigate now the small time limit $|t| \to 0$ for Lorentzian solutions. As we shall see, it corresponds to the vanishing volume limit $V \to 0$ and takes place for $E \ge 0$ if $B, C > 0$ or $B > 0, C < 0$ and for $E > 0$ if $B, C < 0$ or $B < 0, C > 0$ (see Fig. 7 and Fig. 8). First, we consider the case of positive energies $E > 0$. As follows from (7.194), (7.217), (7.236), and (7.240)

$$
\exp\left(\sqrt{2E}q\tau\right) \sim t^{-1}, \quad t \to 0 \qquad (7.242)
$$

and with the help of equations (7.153), (7.155), and (7.156) we obtain for the scale factors in this limit the expressions

$$a_i \approx \bar{A}_i t^{\bar{\alpha}^i}, \quad t \to 0, \tag{7.243}$$

where

$$\bar{\alpha}^i = \frac{1}{D-1} - \frac{1}{\sqrt{2E}q}\alpha^i \tag{7.244}$$

and the parameters satisfy the conditions

$$\sum_{i=1}^{n} d_i \bar{\alpha}^i = 1, \tag{7.245}$$

$$\sum_{i=1}^{n} d_i (\bar{\alpha}^i)^2 = 1 - \frac{1}{q^2} \frac{\kappa^2 A^{(1)}}{\varepsilon + \kappa^2 A^{(1)}} \to 1 \quad \text{for} \quad A^{(1)} \to 0 \tag{7.246}$$

in accordance with the Eqs. (7.176) to (7.178). For the volume of the universe, we obtain in this limit

$$V \sim t, \quad t \to 0. \tag{7.247}$$

Thus, for positive energy, $E > 0$, and small synchronous times the universe behaves like the Kasner universe.

Now, we consider the exceptional case $E = 0$. It follows from (7.193) that

$$\tau \approx \frac{2}{q^2 B} \frac{1}{t}, \quad t \to 0 \tag{7.248}$$

and for the volume, we obtain from (7.196)

$$V \sim t^2, \quad t \to 0. \tag{7.249}$$

With the help of Eq. (7.153), we conclude that the approximation of the scale factors is given by

$$a_i \approx \bar{A}_i t^{2/(D-1)} \exp\left(\frac{2\alpha_i}{q^2 B}\frac{1}{t}\right), \quad t \to 0 \tag{7.250}$$

in accordance with expression (7.180) for $\alpha^{(2)} = 1$.

Thus, the scale factors behave either anisotropically and exponentially like

$$a_i \sim \exp\left(\frac{2\alpha_i}{q^2 B}\frac{1}{t}\right), \quad t \to 0 \tag{7.251}$$

if $\alpha_i \neq 0 \quad (\varepsilon > 0,\ A^{(1)} < 0)$ or they have power law behaviour like

$$a_i \sim t^{2/(D-1)}, \quad t \to 0 \tag{7.252}$$

if $\alpha_i = 0 \quad (\varepsilon = 0,\ A^{(1)} = 0)$ (see Eq. (7.181) for $\alpha^{(2)} = 1$). In the latter case, the free minimally coupled scalar field is absent ($A^{(1)} = 0$).

Similar investigations can be done for the equation (7.232) shifted in time such that $V \approx t^2, \quad t \to 0$.

If $E < 0$, the universe has in the Lorentzian region a classical turning point at the minimal volume V_{min} and reaches never $V = 0$ (see Fig. 7 and Fig. 8).

Now, let us consider the infinite volume limit $V \to 0$ which, as we shall see, coincides with the limit $t \to +\infty$. As follows from Fig. 7 and Fig. 8, this is possible if B, $C > 0$ or $B < 0$, $C > 0$. With Eqs. (7.193) to (7.195) and (7.216), (7.218) to (7.220) one can demonstrate that $\tau \to$ *constant* if $t \to +\infty$ and the constant can be put equal to zero (with a proper choice of the integration constant τ_0). From (7.153) follows that isotropization takes place in this limit, namely

$$a_i \sim V^{1/D-1}, \quad t \to +\infty \tag{7.253}$$

and from Eqs. (7.196) to (7.198) and (7.221), (7.223) to (7.225) we get

$$V \approx \exp\left(\sqrt{2C}qt\right), \quad t \to +\infty \tag{7.254}$$

in accordance with (7.169).

Thus, if $C > 0$ we obtain in the limit $t \to +\infty$ an (isotropic) de Sitter universe. If $C < 0$, the universe has a classical turning point at maximal volume V_{max} and the volume can not go to infinity.

Let us come back once more to the case $C > 0$ describing a universe arising from "nothing". The volume is given by (7.201), (7.225), and (7.223) and the harmonic time coordinate and the synchronous one are related via (7.200), (7.220), and (7.218), respectively. We shall restrict ourselves to the case $E < 0$ for simplicity. In this case, we get the asymptotic expression

$$\tau \approx \frac{1}{\sqrt{|E|/2q}} \arctan \frac{\sqrt{2|E|C}}{\sqrt{|\Delta|} - B} \equiv A, \tag{7.255}$$

if $t \gg \left(\sqrt{\frac{C}{2}}q\right)^{-1}$ (it is sufficient to satisfy $\sqrt{\frac{C}{2}}qt \geq 2$). Then, as follows from equation (7.153), the scale factors are given by

$$a_i \approx A_i \exp(\alpha^i A) V^{1/(D-1)}. \tag{7.256}$$

In [122] was shown that for $4 \lesssim \sqrt{\frac{C}{2}}qt \ll D - 1$ the parameters of the model can be chosen such that, due to the exponential prefactor in (7.256), some of the factor spaces (with $\alpha^i > 0$) undergo inflation after birth from "nothing" while other factor spaces (with $\alpha^i < 0$) remain compactified near the Planck length $L_{\mathrm{Pl}} \approx 10^{-33} cm$. The (graceful exit) mechanism responsible for the transition from the inflationary stage to the Kasner-like stage, in which the scale factors of the external spaces M_i exhibit power-law behaviour while the internal spaces remain frozen in near the Planck scale, deserves still more detailed investigations, similar those of [122]. (There the complementary case of multidimensional cosmological models with cosmological constant was considered.)

7.5.4 Classical wormholes

In this section we consider in more detail a special type of instantons, called wormholes. These usually are Riemannian metrics, consisting of two large regions joined by a narrow throat (handle). Obviously, they appear if the classical Lorentzian solutions of the model have turning points at some maximum, Namely, according to Fig. 7 and 8, for models with $C < 0$ (the parameter B may be positive as well as negative). Let us show this explicitly. We consider instantons which can be obtained by analytic continuation $t \to -it$ of the Lorentzian solutions (7.232), (7.233), (7.238) and (7.241) respectively.

$$V = \frac{B}{|C|}\cosh^2\left(\sqrt{|C|/2}qt\right), \quad E=0, \quad |t|<\infty, \tag{7.257}$$

$$V = \frac{\cosh^2\left(\sqrt{|C|/2}qt\right)}{2|C|}\left[\sqrt{|\Delta|}+B+\left(\sqrt{|\Delta|}-B\right)\tanh^2\left(\sqrt{|C|/2}qt\right)\right], \tag{7.258}$$
$$E>0, \quad |t|<\infty,$$

$$V = \frac{\cosh^2\left(\sqrt{|C|/2}qt\right)}{2|C|}\left[B+\sqrt{|\Delta|}-\left(B-\sqrt{|\Delta|}\right)\tanh^2\left(\sqrt{|C|/2}qt\right)\right], \tag{7.259}$$
$$-B^2/4|C|<E<0, \quad |t|<\infty,$$

$$V = \frac{\cosh^2\left(\sqrt{|C|/2}qt\right)}{2|C|}\left[\sqrt{|\Delta|}-|B|+\left(\sqrt{|\Delta|}+|B|\right)\tanh^2\left(\sqrt{|C|/2}qt\right)\right], \tag{7.260}$$
$$E>0, \quad |t|<\infty.$$

As mentioned before, these equations correspond (with evident substitutions) to the Lorentzian equations (7.221), (7.225), (7.223) and (7.201), respectively. The harmonic and synchronous times are related respectively by

$$\tau = \frac{\sqrt{2|C|}}{Bq}\tanh\left(\sqrt{|C|/2}qt\right), \qquad E=0, \tag{7.261}$$

$$\tan\left(\sqrt{E/2}q\tau\right) = \frac{\sqrt{4E|C|}}{\sqrt{|\Delta|}+B}\tanh\left(\sqrt{|C|/2}qt\right), \qquad E>0, \tag{7.262}$$

$$\tanh\left(\sqrt{|E|/2}q\tau\right) = \frac{\sqrt{|\Delta|}-B}{\sqrt{4EC}}\tanh\left(\sqrt{|C|/2}qt\right), \qquad -\frac{B^2}{4|C|}<E<0, \tag{7.263}$$

$$\tan\left(\sqrt{E/2}q\tau\right) = \frac{\sqrt{4E|C|}}{\sqrt{|\Delta|}-|B|}\tanh\left(\sqrt{|C|/2}qt\right), \qquad E>0. \tag{7.264}$$

(See (7.216), (7.220), (7.218) and (7.200) respectively. (7.218) looks like (7.263), if we choose the constant of integration τ_0 such that $f|_{\tau=0} = V_0^{(2)}$, whence $f = \sqrt{|\Delta|}\exp\left(\sqrt{2E}q\tau\right)$, and use the relation $f = |B| - \sqrt{4EC}\tanh\left[\sqrt{C/2}qt + \operatorname{artanh}\frac{|B|-\sqrt{|\Delta|}}{\sqrt{4EC}}\right]$, where a turning point appears for $t=0$.)

It can easily be seen from (7.261) to (7.264) that the harmonic time τ is finite for the full range $-\infty < t < \infty$ and goes to constants when $|t| \to +\infty$.

For the spatial volume of the universe we have the asymptotic behaviour

$$V \sim \exp\left(\sqrt{2|C|}\, q\, |t|\right), \quad |t| \to \infty, \tag{7.265}$$

for all cases (7.257) to (7.260).

In the Euclidean region (7.153) holds unchanged, since the Wick rotation $\tau \to -i\tau$ has to be accompanied by the transformation $\alpha^j \to i\alpha^j$ ($p^j \to ip^j$). This means that the parameter ε in the constraint equation (7.149) remains unchanged (see (7.148)).

Thus, the Euclidean metric reads

$$ds^2 = dt^2 + a_1^2(t) g^{(1)} + \ldots + a_n^2(t) g^{(n)}, \tag{7.266}$$

where each scale factor a_i has its own turning point at "time" t_i, when $\frac{d}{dt} a_i = 0$. The metric has its most simple and symmetric form in the case $\varepsilon = 0$ ($\alpha^i = 0$, $i = 1, \ldots, n$), whence

$$ds^2 = dt^2 + V^{\frac{2}{D-1}} \left(g^{(1)} + \ldots + g^{(n)}\right), \tag{7.267}$$

where V is given by equations (7.257) to (7.260), and the throat is located at $t = 0$. In the limit $|t| \to \infty$, the metric (7.266) converges to

$$ds^2 = dt^2 + \exp\left(\frac{2\sqrt{2|C|}}{D-1}\, q\, |t|\right) \left(g^{(1)} + \ldots + g^{(n)}\right), \tag{7.268}$$

which describes an anti-de Sitter Euclidean universe. Thus, the metric (7.266) describes asymptotically anti-de Sitter wormholes.

The structure of a universe for models with classical Euclidean wormholes is shown schematically in Fig. 11 und Fig. 12 for the symmetric case $\alpha^i = 0$ ($i = 1, \ldots, n$) with a metric (7.267). There are two qualitatively different pictures. The first case (see Fig. 11) takes place for $E \geq 0$ ($B > 0$, $C < 0$) and for $E > 0$ ($B, C < 0$) and describes asymptotically an anti-de Sitter wormhole and a baby universe which can branch off from this wormhole. The second case (see Fig. 12) takes place for $-B^2/4|C| < E < 0$ ($B > 0$, $C < 0$) and describes, besides wormhole and baby universe, an additional parent instanton which is responsible for the birth of the universe from nothing.

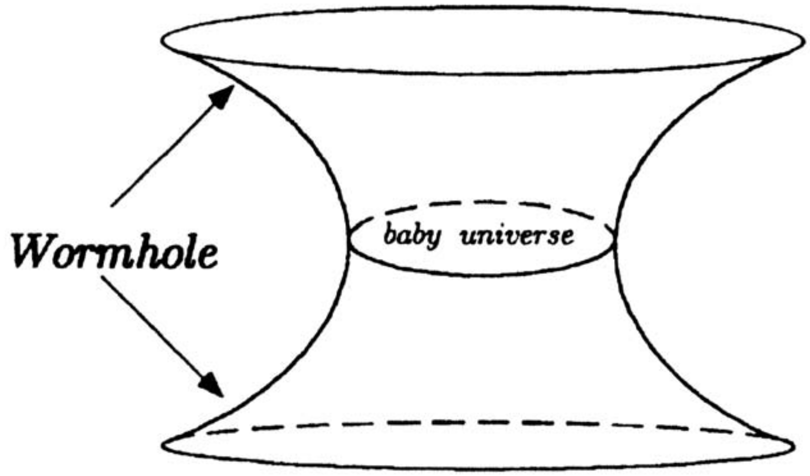

Figure 11: An asymptotically anti-de Sitter wormhole is shown schematically for energies $E \geq 0$ $(B > 0, C < 0)$ and $E \geq 0$ $(B, C < 0)$ in the symmetrical case $\alpha^i = 0$ $(i = 1, \ldots, n)$.

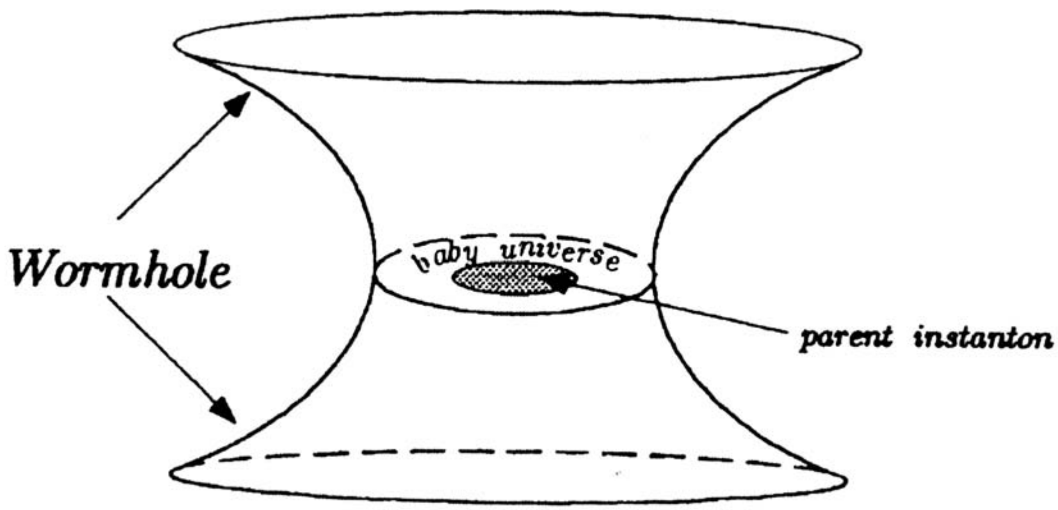

Figure 12: The qualitative structure of the universe is is shown schematically for energies $-U_m < E < 0$ $(B > 0,\, C < 0)$ in the symmetrical case $\alpha^i = 0$ $(i = 1, \ldots, n)$.

7.5.5 Reconstruction of potentials

The effective perfect fluid Lagrangian (7.133) has its origin in the scalar field Lagrangian (7.128). Below we calculate a class of potentials $U^{(a)}(\varphi^{(a)})$ which ensure the equivalence of these Lagrangians. The procedure of potential reconstruction was proposed in [121] and is applied in the following. For the integrable 3-component model holds

$$\varphi^{(a)} = \pm\frac{\sqrt{\alpha^{(a)}/2}}{q}\int\frac{\sqrt{\rho^{(a)}(V)}dV}{\left[\varepsilon+\kappa^2V^2\left(\rho^{(1)}+\rho^{(2)}+\rho^{(3)}\right)\right]^{1/2}}+\varphi_0^{(a)}, \quad a=1,2,3, \tag{7.269}$$

where the energy density $\rho^{(a)}$ is given by (7.134) and $\alpha^{(1)}=2$, $\alpha^{(2)}=1$, $\alpha^{(3)}=0$. $\varphi_0^{(a)}$ is a constant of integration. We should stress that equation (7.269) was obtained for Lorentzian regions. As a result, we get the scalar fields $\varphi^{(a)}$ as a function of the spatial volume V. Inverting this expression, we find the spatial volume as a function of the scalar field $\varphi^{(a)}$, $V=V(\varphi^{(a)})$, and consequently, a dependence of the energy density $\rho^{(a)}$ on the scalar field $\varphi^{(a)}$, $\rho^{(a)}=\rho^{(a)}(\varphi^{(a)})$. Then, using Eqs. (7.130) to (7.132), we find the potential $U^{(a)}(\varphi^{(a)})$ in the form

$$U^{(a)}(\varphi^{(a)})=\frac{1}{2}\left(2-\alpha^{(a)}\right)\rho^{(a)}(\varphi^{(a)}), \quad a=1,2,3, \tag{7.270}$$

where

$$\rho^{(a)}=A^{(a)}\left[V(\varphi^{(a)})\right]^{-\alpha^{(a)}}. \tag{7.271}$$

The third component of the scalar field has $\alpha^{(3)}=0$. Then, from (7.270) and (7.271) it follows that $\varphi^{(3)}$, $U^{(3)}$, and $\rho^{(3)}$ are constant. This scalar field component with the equation of state $P^{(3)}=-\rho^{(3)}$ is equivalent to the cosmological constant $\Lambda\equiv\kappa^2U^{(3)}=\kappa^2A^{(3)}=C$. For $\alpha^{(1)}=2$, we have $U^{(1)}\equiv 0$ (free scalar field). In this case, the scalar field $\varphi^{(1)}$ is equivalent to a ultra-stiff perfect fluid ($P^{(1)}=\rho^{(1)}$). Equation (7.269) reads in this case

$$\varphi^{(1)}-\varphi_0^{(1)}=\pm\frac{\sqrt{A^{(1)}}}{q}\int\frac{dV}{V\sqrt{E+BV+CV^2}}, \tag{7.272}$$

where E and B are defined by (7.184) and (7.185) respectively.

A consequence of (7.186) is

$$\varphi^{(1)}-\varphi_0^{(1)}=\pm\sqrt{2A^{(1)}}\tau. \tag{7.273}$$

This result is expected for a free minimal coupled scalar field in the harmonic time gauge where $\ddot{\varphi}^{(1)}=0$. After integration in (7.272),

$$\varphi^{(1)}-\varphi_0^{(1)}=\mp i\,\frac{2\sqrt{|A^{(1)}|}}{q}\frac{\sqrt{BV+CV^2}}{BV}, \qquad E=0, \tag{7.274}$$

$$\varphi^{(1)}-\varphi_0^{(1)}=\pm\frac{\sqrt{A^{(1)}}}{\sqrt{E}q}\ln\frac{2E+BV-2\sqrt{ER}}{2V}, \qquad E>0, \tag{7.275}$$

$$\varphi^{(1)}-\varphi_0^{(1)}=\pm i\,\frac{\sqrt{|A^{(1)}|}}{\sqrt{|E|}q}\arcsin\frac{BV-2|E|}{V\sqrt{|\Delta|}}, \qquad E<0, \quad B^2-4EC>0, \tag{7.276}$$

with $R := E + BV + CV^2$ and $|\Delta| = B^2 - 4EC$. For $E \leq 0$ (i.e. $A^{(1)} < 0$) this scalar field is imaginary.

Let us now consider the second component with $\alpha^{(2)} = 1$. The scalar field $\varphi^{(2)}$ is equivalent to dust ($P^{(2)} = 0$). Equation (7.269) reads now

$$\varphi^{(2)} - \varphi_0^{(2)} = \pm \frac{\sqrt{A^{(2)}}}{\sqrt{2}q} \int \frac{dV}{\sqrt{V}\sqrt{E + BV + CV^2}}. \tag{7.277}$$

$\varphi^{(2)}$ is imaginary for $A^{(2)} < 0$, i.e. $B < 0$. The integral in (7.277) is an elliptic one and, in general, it is not possible to express it by elementary functions. But in the particular case $E = 0$, which expresses the asymptotic behaviour of the scalar field (7.277), we get

$$\varphi^{(2)} - \varphi_0^{(2)} = \mp \frac{\sqrt{2}}{\kappa q} \text{arcoth} \left(1 + \frac{C}{B} V\right)^{1/2}, \qquad B, C > 0, \tag{7.278}$$

$$\varphi^{(2)} - \varphi_0^{(2)} = \mp \frac{\sqrt{2}}{\kappa q} \text{artanh} \left(1 - \frac{|C|}{B} V\right)^{1/2}, \qquad B > 0,\ C < 0, \tag{7.279}$$

$$\varphi^{(2)} - \varphi_0^{(2)} = \mp i \frac{\sqrt{2}}{\kappa q} \arctan \left(\frac{C}{|B|} V - 1\right)^{1/2}, \qquad B < 0,\ C > 0, \tag{7.280}$$

the volume of the universe

$$V = \frac{B}{C} \sinh^{-2} \left[\frac{\kappa q}{\sqrt{2}} \left(\varphi^{(2)} - \varphi_0^{(2)}\right)\right], \qquad B, C > 0, \tag{7.281}$$

$$V = \frac{B}{|C|} \cosh^{-2} \left[\frac{\kappa q}{\sqrt{2}} \left(\varphi^{(2)} - \varphi_0^{(2)}\right)\right], \qquad B > 0,\ C < 0, \tag{7.282}$$

$$V = \frac{|B|}{C} \cos^{-2} \left[\frac{\kappa q}{\sqrt{2}} i \left(\varphi^{(2)} - \varphi_0^{(2)}\right)\right], \qquad B < 0,\ C > 0, \tag{7.283}$$

and the potential of the scalar field

$$U^{(2)}(\varphi^{(2)}) = \frac{C}{2\kappa^2} \sinh^2 \left[\frac{\kappa q}{\sqrt{2}} \left(\varphi^{(2)} - \varphi_0^{(2)}\right)\right], \qquad B, C > 0, \tag{7.284}$$

$$U^{(2)}(\varphi^{(2)}) = \frac{|C|}{2\kappa^2} \cosh^2 \left[\frac{\kappa q}{\sqrt{2}} \left(\varphi^{(2)} - \varphi_0^{(2)}\right)\right], \qquad B > 0,\ C < 0, \tag{7.285}$$

$$U^{(2)}(\varphi^{(2)}) = \frac{C}{2\kappa^2} \cos^2 \left[\frac{\kappa q}{\sqrt{2}} i \left(\varphi^{(2)} - \varphi_0^{(2)}\right)\right], \qquad B < 0,\ C > 0. \tag{7.286}$$

It follows from these equations that for $B, C > 0$ and $B, C < 0$ the volume goes to infinity like

$$V \sim \frac{1}{\left|\varphi^{(2)}\right|^2} \to +\infty, \quad \left|\varphi^{(2)}\right| \to 0. \tag{7.287}$$

The general expression (7.277) should have the same asymptotic behaviour in all the cases where the limit $V \to +\infty$ is permitted, because we can drop in this limit the term E in the denominator of (7.277).

If $E > 0$, from (7.277) results

$$\varphi^{(2)} - \varphi_0^{(2)} \approx \frac{\sqrt{2A^{(2)}/E}}{q} \sqrt{V} \to 0, \quad V \to 0. \tag{7.288}$$

Let us now consider two particular cases of (7.277) for $E \neq 0$. The first case is that one when the classical trajectory has two turning points $V_0^{(1,2)}$, i. e. when either $B > 0$, $C < 0$ or $B < 0$, $C > 0$ (for both the cases $B^2 > 4EC$). Then (see equation (3.131) in [136]),

$$\varphi^{(2)} = \pm \frac{2}{\kappa q} \frac{\sqrt{B}}{\sqrt{|B| + \sqrt{|\Delta|}}} F(\psi|m) \tag{7.289}$$

where $F(\psi|m)$ is the elliptic integral of the first kind [137] and

$$\psi = \arcsin\sqrt{(V_0^{(2)} - V)/(V_0^{(2)} - V_0^{(1)})}, \qquad V_0^{(2)} > V \geq V_0^{(1)}, \quad B > 0, C < 0, \tag{7.290}$$

$$\psi = \arcsin\sqrt{V/V_0^{(1)}}, \qquad V_0^{(2)} > V_0^{(1)} \geq V, \quad B < 0,\ C > 0, \tag{7.291}$$

$$\psi = \arcsin\sqrt{(V - V_0^{(2)})/(V - V_0^{(1)})}, \qquad V > V_0^{(2)} > V_0^{(1)}, \quad B < 0,\ C > 0 \tag{7.292}$$

$$m = \sqrt{1 - V_0^{(1)}/V_0^{(2)}}, \qquad V_0^{(2)} > V \geq V_0^{(1)}, \quad B > 0,\ C < 0, \tag{7.293}$$

$$m = \sqrt{V_0^{(1)}/V_0^{(2)}}, \qquad V_0^{(2)} > V_0^{(1)} \geq V, \quad B < 0,\ C > 0, \tag{7.294}$$

$$m = \sqrt{V_0^{(1)}/V_0^{(2)}}, \qquad V > V_0^{(2)} > V_0^{(1)}, \quad B < 0,\ C > 0. \tag{7.295}$$

The turning points are

$$V_0^{(1,2)} = \frac{|B| \mp \sqrt{|\Delta|}}{2|C|}. \tag{7.296}$$

The minus sign is related to $V_0^{(1)}$, the plus sign to $V_0^{(2)}$, and $|\Delta| = B^2 - 4EC$. The scalar field $\varphi^{(2)}$ is imaginary for $B < 0$.

With the Jacobian elliptic functions [137], inverting (7.289), the volume of the universe is given by

$$V = V_0^{(2)} - \mathrm{sn}^2(V_0^{(2)} - V_0^{(1)}), \qquad V_0^{(2)} > V \geq V_0(1), \quad B > 0,\ C < 0, \tag{7.297}$$

$$V = \mathrm{sn}^2 V_0^{(1)}, \qquad V_0^{(2)} > V_0^{(1)} \geq V, \quad B < 0,\ C > 0, \tag{7.298}$$

$$V = \frac{V_0^{(2)} - \mathrm{sn}^2 V_0^{(1)}}{1 - \mathrm{sn}^2}, \qquad V > V_0^{(2)} > V_0^{(1)}, \quad B < 0,\ C > 0, \tag{7.299}$$

where $\mathrm{sn} \equiv \mathrm{sn}\left(I_1\varphi^{(2)}|m\right) = \sin\psi$ and $I_1^{-1} = \pm\frac{2\sqrt{B}}{\kappa q\sqrt{|B|+|\Delta|}}$. The corresponding potential terms are then given as $U^{(2)} = A^{(2)}/2V$ (see (7.270) and (7.271) for $\alpha^{(2)} = 1$). According to the properties of the Jacobian elliptic functions [137] asymptotic estimates for (7.298) and (7.299) are $V \approx (q^2E/2A^{(2)})(\varphi^{(2)})^2$ for $|\varphi^{(2)}| \to 0$ (in accordance with (7.288)) and $V \sim 1/|\varphi^{(2)}|^2$ for $|\varphi^{(2)}| \to 0$ (in accordance with (7.287)).

Another particular case for $C > 0$ is that with $E > B^2/4C$. Here (see (3.138 (7)) in [136]), we obtain

$$\varphi^{(2)} = \pm \frac{\sqrt{B/2C}}{\kappa q} \frac{1}{(E/C)^{1/4}} F(\psi|m), \tag{7.300}$$

where

$$\psi = 2\arctan\sqrt{V/\sqrt{E/C}} \tag{7.301}$$

and

$$m=\sqrt{(2\sqrt{EC}-B)/4\sqrt{EC}}. \tag{7.302}$$

Inverting equation (7.300), we obtain

$$V=\frac{2-\mathrm{sn}^2}{\mathrm{sn}^2}\sqrt{E/C}\pm\sqrt{\left(\frac{2-\mathrm{sn}^2}{\mathrm{sn}^2}\sqrt{E/C}\right)^2-\frac{E}{C}} \tag{7.303}$$

with $\mathrm{sn}\equiv\mathrm{sn}(I_2\varphi^{(2)}|m)=\sin\psi$ and $I_2^{-1}=\pm\frac{\sqrt{B/2C}}{\kappa q(E/C)^{1/4}}$. For the branch with the plus sign, $V\sim\frac{1}{|\varphi^{(2)}|^2}\to\infty$ for $|\varphi^{(2)}|\to 0$ (in accordance with (7.287)) and for the branch with the minus sign, $V\approx(q^2E/2A^{(2)})(\varphi^{(2)})^2\to 0$ for $|\varphi^{(2)}|\to 0$ (in accordance with (7.288)). To find the scalar field potential, we have to substitute (7.303) into $U^{(2)}(\varphi^{(2)})=A^{(2)}/2V$.

7.5.6 Solutions to the quantized model

At the quantum level, the constraint equation (7.145) is replaced by the Wheeler-DeWitt (WDW) equation. The WDW equation is covariant with respect to gauge as well as minisuperspace coordinate transformations [38]. In the harmonic time gauge [118], [38] it reads

$$\left(\frac{1}{2}\frac{\partial^2}{\partial z^{0^2}}-\frac{1}{2}\sum_{i=1}^{n-1}\frac{\partial^2}{\partial z^{i^2}}+\kappa^2\sum_{a=1}^{m}A^{(a)}\exp(k^{(a)}qz^0)\right)\Psi=0. \tag{7.304}$$

Formally, this WDW equation has the same structure as that one in [124]. So, we can take over some results of [124]. However, on the semiclassical level the dynamics of the universe is quite different for the models in both the papers. Semiclassical equations were considered in [91].

We look for solutions of (7.304) by separation of the variables and try the ansatz

$$\Psi(z)=\Phi(z^0)\exp\left(i\mathbf{p}\cdot\mathbf{z}\right), \tag{7.305}$$

where $\mathbf{p}:=(p^1,\dots,p^{n-1})$ is a constant vector, $\mathbf{z}:=(z^1,\dots,z^{n-1})$, $p_i=p^i$ and $\mathbf{p}\cdot\mathbf{z}:=\sum_{i=1}^{n-1}p_iz^i$. Substitution of (7.305) into (7.304) gives

$$\left[\frac{1}{2}\frac{d^2}{dz^{0^2}}+\frac{1}{2}\sum_{i=1}^{n-1}(p_i)^2+\kappa^2\sum_{a=1}^{m}A^{(a)}\exp\left(k^{(a)}qz^0\right)\right]\Phi=0. \tag{7.306}$$

For the integrable 3-component model this equation reduces to

$$\left[-\frac{1}{2}\frac{d^2}{dz^{0^2}}+U(z^0)\right]\Phi=E\Phi \tag{7.307}$$

in the notation of (7.184) and (7.185). Following [124], we rewrite this equation like

$$x^2\frac{d^2\Phi}{dx^2}+x\frac{d\Phi}{dx}+\left[\tilde{E}+\tilde{B}x+\tilde{C}x^2\right]\Phi=0, \tag{7.308}$$

where $\tilde{E}=2E/q^2$, $\tilde{B}=2B/q^2$, $\tilde{C}=2C/q^2$, and $x=\exp(qz^0)$ (x is identical to the spatial volume of the universe V). (7.308) is equivalent to the Whittaker equation

$$\frac{d^2y}{d\xi^2}+\left[-\frac{1}{4}+\frac{\tilde{B}/T}{\xi}+\frac{\tilde{E}+1/4}{\xi^2}\right]y=0, \tag{7.309}$$

where $T:=\pm 2i\sqrt{\tilde{C}}$, $\xi:=Tx$, and $\Phi=:x^{-1/2}y(\xi)$ and also equivalent to the Kummer equation

$$\xi\frac{d^2w}{d\xi^2}+(1+2\mu-\xi)\frac{dw}{d\xi}-\left[\frac{1}{2}+\mu-\frac{\tilde{B}}{T}\right]w=0, \tag{7.310}$$

where $\mu^2:=-\tilde{E}$ and $\Phi=:x^{-1/2}\exp\left(-\frac{1}{2}\xi\right)\xi^{\frac{1}{2}+\mu}w(\xi)$. In the first case, the solutions are the Wittaker functions [137] $y_1:=M_{k,\mu}(\xi)$ and $y_2:=W_{k,\mu}(\xi)$ with $k:=\tilde{B}/T$ and $\mu^2:=-\tilde{E}$. In the second case, the solutions are the Kummer functions [137] $w_1:=M(a,b,\xi)$ and $w_2:=U(a,b,\xi)$ with $a:=\frac{1}{2}+\mu-\tilde{B}/T$ and $b:=1+2\mu$.

The general solution of equation (7.304) for the 3-component model is

$$\Psi(z)=\sum_{i=1,2}\int d^{n-1}p\,C_i(\mathbf{p})\exp\left(i\mathbf{p}\cdot\mathbf{z}\right)\,\Phi_E^{(i)}(\exp(qz^0)), \tag{7.311}$$

where $\Phi_E^{(1,2)}=\frac{1}{\sqrt{x}}y_{1,2}(\xi)$, or $\Phi_E^{(1,2)}=\frac{1}{\sqrt{x}}\exp\left(-\frac{1}{2}\xi\right)\xi^{\frac{1}{2}+\mu}w_{1,2}(\xi)$. It is convenient to set $T=+2i\sqrt{\tilde{C}}$ for $C>0$ and $T=-2i\sqrt{\tilde{C}}$ for $C<0$, and $\mu:=+\sqrt{-\tilde{E}}$.

In [91], it was argued that the parameter E can be interpreted as energy. So, the state with $E=0$, vanishing momenta p_i ($i=1,\ldots,n-1$), and $A^{(1)}=0$ (absence of free scalar field excitations) is the ground state of the system. Thus, its wave function reads

$$\Psi_0=\Phi_0^{(i)}\left(\exp(qz^0)\right), \quad i=1,2. \tag{7.312}$$

The limit of large spatial geometries in (7.308) $z^0\to+\infty$ (remember $x\equiv V=\exp(qz^0)$) is equivalent to $\tilde{B}\to 0$. In this limit, the Wittaker functions reduce to Bessel functions [137], namely

$$M_{k,\mu}(\xi)\underset{k\to 0}{\longrightarrow}\sqrt{V}J_\mu\left(\sqrt{\tilde{C}}V\right), \tag{7.313}$$

$$W_{k,\mu}(\xi)\underset{k\to 0}{\longrightarrow}\sqrt{V}H_\mu^{(2)}\left(\sqrt{\tilde{C}}V\right) \tag{7.314}$$

for $C>0$ and

$$M_{k,\mu}(\xi)\underset{k\to 0}{\longrightarrow}\sqrt{V}I_\mu\left(\sqrt{|\tilde{C}|}V\right), \tag{7.315}$$

$$W_{k,\mu}(\xi)\underset{k\to 0}{\longrightarrow}\sqrt{V}K_\mu\left(\sqrt{|\tilde{C}|}V\right) \tag{7.316}$$

for $C<0$.

Following the ideas of [91, 138], one can demonstrate that for $C > 0$ the wave function

$$\Psi_0^{HH} = \Phi_0^{(1)} \underset{k\to 0}{\longrightarrow} J_0\left(\frac{\sqrt{2C}}{q}V\right) \underset{V\to\infty}{\sim} \cos\left(\frac{\sqrt{2C}}{q}V\right) \tag{7.317}$$

corresponds to the Hartle-Hawking boundary condition [139] and the wave function

$$\Psi_0^{V} = \Phi_0^{(2)} \underset{k\to 0}{\longrightarrow} H_0^{(2)}\left(\frac{\sqrt{2C}}{q}V\right) \underset{V\to\infty}{\sim} \exp\left(-i\frac{\sqrt{2C}}{q}V\right) \tag{7.318}$$

corresponds to the Vilenkin boundary condition [140].

In the case $C < 0$, we get

$$\Psi_0^{HH} = \Phi_0^{(1)} \underset{k\to 0}{\longrightarrow} I_0\left(\frac{\sqrt{2|C|}}{q}V\right) \underset{V\to\infty}{\sim} \exp\left(\frac{\sqrt{2|C|}}{q}V\right) \tag{7.319}$$

The potential has for $B > 0$, $C < 0$ as well and for $E < 0$ the energy spectrum is discrete (see Fig. 8). In this case, the finite solutions of the wave equation (7.307) [141] are

$$\Phi_n = \exp\left(-\frac{1}{2}\xi\right)\xi^{\mu} M(-n, b, \xi). \tag{7.320}$$

The energy levels are given by

$$-E_n = \left[\frac{B}{2\sqrt{|C|}} - \frac{q}{\sqrt{2}}\left(n + \frac{1}{2}\right)\right]^2. \tag{7.321}$$

n is a non negative integer and restricted to

$$n < \frac{B}{q\sqrt{2|C|}} - \frac{1}{2}. \tag{7.322}$$

Thus, the discrete spectrum has a finite number of eigenvalues. If $\frac{B}{q\sqrt{2|C|}} < \frac{1}{2}$, there is no discrete spectrum. It was demonstrated in [124] that the wave functions (7.320) satisfy the quantum wormhole boundary conditions [142].

7.6 The Einstein frame in cosmology

In this section we discuss the issue of the physical frame for the particularly important case $D_0 = 4$. First of all, in this case a self-dual canonical formulation of dynamics is at hand, due to the particular spinor decomposition $\mathrm{so}(1,3) = \mathrm{su}(2) \oplus \mathrm{su}(2)$ of the tangent Lorentz symmetry. In the Einstein frame, the effective σ-model with $D_0 = 4$ admits in principle a canonical quantization of the geometry on $\overline{M}_0 = \mathrm{I\!R} \times M_0$ to the same extend and under the same assumptions as pure Einstein gravity does.

Since for a multidimensional geometry as defined above the imprint of the internal factor spaces is only by their scale factors, configuration space and phase space of such geometries will only be extended by a finite number of dilatonic midisuperspace fields.

However, only in the Einstein frame the coupling of the dilatonic fields to the $\overline{D}_0$-geometry will be minimal such that the quantization of the latter can be executed practically independently.

In the following we denote the external space-time metric $\overline{g}^{(0)}$ in the Brans-Dicke frame with $\overline{\gamma} \stackrel{!}{=} 0$ as $\overline{g}^{(\mathrm{BD})}$ and in the Einstein frame with $f \stackrel{!}{=} 0$ as $\overline{g}^{(\mathrm{E})}$. It can be easily seen that they are connected with each other by a conformal transformation

$$\overline{g}^{(\mathrm{E})} \mapsto \overline{g}^{(\mathrm{BD})} = \Omega^2 \overline{g}^{(\mathrm{E})} \tag{7.323}$$

with Ω from (6.39).

In particular, also for spatially homogeneous cosmological models (and with $t \leftrightarrow u$ for spherically symmetric static models) solutions have to be transformed to the Einstein frame before a physical interpretation can be given.

7.6.1 Generating solutions in the Einstein frame

Let us now consider the space time foliation $\overline{M}_0 = \mathrm{I\!R} \times M_0$ where $g^{(0)}$ is a smooth homogeneous metric on M_0. Under any projection $\mathrm{pr}_0 : \overline{M_0} \to \mathrm{I\!R}$ a consistent pullback of the metric $-e^{2\gamma(\tau)} d\tau \otimes d\tau$ from $\tau \in \mathrm{I\!R}$ to $x \in \mathrm{pr}_0^{-1}\{\tau\} \subset \overline{M}_0$ is given by

$$\overline{g}^{(\mathrm{BD})}(x) := -e^{2\gamma(\tau)} d\tau \otimes d\tau + e^{2\beta^0(x)} g^{(0)}. \tag{7.324}$$

For spatially (metrically-)homogeneous cosmological models as considered below all scale factors $a_i := e^{\beta^i}$, $i = 0, \ldots, n$, depend only on $\tau \in \mathrm{I\!R}$.

With (7.324) and (7.323), the multidimensional metric reads

$$\begin{aligned} g &= -e^{2\gamma(\tau)} d\tau \otimes d\tau + a_0^2 g^{(0)} + \sum_{i=1}^{n} e^{2\beta^i} g^{(i)} \\ &= -dt_{\mathrm{BD}} \otimes dt_{\mathrm{BD}} + a_{\mathrm{BD}}^2 g^{(0)} + \sum_{i=1}^{n} e^{2\beta^i} g^{(i)} \\ &= -\Omega^2 dt_{\mathrm{E}} \otimes dt_{\mathrm{E}} + \Omega^2 a_{\mathrm{E}}^2 g^{(0)} + \sum_{i=1}^{n} e^{2\beta^i} g^{(i)}, \end{aligned} \tag{7.325}$$

where $a_0 := a_{\mathrm{BD}}$ and a_{E} are the external space scale factor functions depending respectively on the cosmic synchronous time t_{BD} and t_{E} in the Brans-Dicke and the Einstein frame. With (6.39) the latter is related to the former by

$$a_{\mathrm{E}} = \Omega^{-1} a_{\mathrm{BD}} = \left(\prod_{i=1}^{n} e^{d_i \beta^i} \right)^{\frac{1}{D_0 - 2}} a_{\mathrm{BD}}, \tag{7.326}$$

and the cosmic time of the Einstein frame is given by

$$\pm dt_{\mathrm{E}} = \Omega^{-1} e^{\gamma} d\tau = \left(\prod_{i=1}^{n} e^{d_i \beta^i} \right)^{\frac{1}{D_0 - 2}} dt_{\mathrm{BD}} \ . \tag{7.327}$$

Since $a_{\mathrm{BD}}^2 (d\eta_{\mathrm{BD}})^2 = \Omega^2 a_{\mathrm{E}}^2 (d\eta_{\mathrm{E}})^2$, the conformal times of the Einstein and the Brans-Dicke frame agree (up to time reversal). This has sometimes guided authors to compare the frames in conformal time (see e.g. [143]). However (at least for cosmology) the physical relevant time is the cosmic synchronous time, which is different for different frames. The presently best known (spatially homogeneous) cosmological solutions were found in the Brans-Dicke frame (see e.g. [144, 119, 124, 30] and an extensive list of references there). Most of them are described most simply within one of the following two systems of target space coordinates. We set

$$q := \sqrt{\frac{D-1}{D-2}} \,, \qquad p := \sqrt{\frac{d_0-1}{d_0}} \,. \tag{7.328}$$

With $\Sigma_k = \sum_{i=k}^{n} d_i$, the first coordinate system [118] is related to β^i, $i = 0, \ldots, n$, as

$$\begin{aligned} z^0 &:= q^{-1} \sum_{j=0}^{n} d_j \beta^j \,, \\ z^i &:= [d_{i-1}/\,\Sigma_{i-1}\Sigma_i]^{1/2} \sum_{j=i}^{n} d_j \left(\beta^j - \beta^{i-1}\right) \,, \quad i = 1, \ldots, n \,, \end{aligned} \tag{7.329}$$

and the second one [138] as

$$\begin{aligned} v^0 &:= p^{-1}(\sum_{j=0}^{n} d_j \beta^j - \beta^0) \,, \\ v^1 &:= p^{-1}[(D-2)/d_0\Sigma_1]^{1/2} \sum_{j=1}^{n} d_j \beta^j, \\ v^i &:= [d_{i-1}/\,\Sigma_{i-1}\Sigma_i]^{1/2} \sum_{j=i}^{n} d_j \left(\beta^j - \beta^{i-1}\right) \,, \quad i = 2, \ldots, n \,, \end{aligned} \tag{7.330}$$

In both of this minisuperspace coordinates the target space Minkowski metric G is given in form of the standard diagonal matrix $G_{ij} := (-)^{\delta_{0i}} \delta_{ij}$. The two coordinates are related by a Lorentz boost in the (01)-plane.

In coordinates (7.329) some known solutions (see e.g. [30, 91, 121]) take the form

$$a_i = A_i (e^{q z^0})^{\frac{1}{D-1}} e^{\alpha^i \tau}, \quad i = 0, \ldots, n, \tag{7.331}$$

where parameters α^i satisfy conditions

$$\begin{aligned} \sum_{i=0}^{n} d_i \alpha^i &= 0, \\ \sum_{i=0}^{n} d_i (\alpha^i)^2 &= 2\epsilon \end{aligned} \tag{7.332}$$

and ϵ is a non-negative parameter.

In coordinates (7.330) some known solutions (see e.g. [124, 145]) take the form

$$\begin{aligned} a_0 &= A_0 (e^{p v^0})^{\frac{1}{d_0-1}} e^{\alpha^0 \tau}, \\ a_i &= A_i e^{\alpha^i \tau}, \quad i = 1, \ldots, n, \end{aligned} \tag{7.333}$$

where parameters α^i satisfy conditions

$$\sum_{i=0}^{n} d_i\alpha^i = \alpha^0, \tag{7.334}$$
$$\sum_{i=0}^{n} d_i(\alpha^i)^2 = (\alpha^0)^2 + 2\epsilon$$

and ϵ is a non-negative parameter.

Explicit expressions for functions $z^0 \equiv z^0(\tau)$ and $v^0 \equiv v^0(\tau)$ depend on the details of the particular models.

Solutions of the form (7.331) and (7.333) were found in the harmonic time gauge $\gamma \overset{!}{=} \sum_{j=0}^{n} d_j\beta^j$, where τ is the harmonic time of the Brans-Dicke frame. Equation (7.329) shows that the coordinate z^0 is related to the dynamical part of the total spatial volume in the BD frame: $v := e^{qz^0} = \prod_{i=0}^{n} {a_i}^{d_i}$.

Relations (7.326) and (7.330) between the different minisuperspace coordinates imply that

$$(d_0-1)\beta^0_E = (d_0-1)\beta^0 + \sum_{j=1}^{n} d_j\beta^j = pv^0\ , \tag{7.335}$$

which shows that the coordinate v^0 is proportional to the logarithmic scale factor of external space in the Einstein frame: $a_E := e^{\beta^0_E}$.

Thus target space coordinates z have the most natural interpretation in the Brans-Dicke frame, whereas target space coordinates v are better adapted to the Einstein frame.

Via (7.335) synchronous time in the Einstein frame is related to harmonic time τ in the Brans-Dicke frame by integration of (7.327) with integration constant c to

$$|t_{\rm E}| + c = \int \left(e^{pv^0}\right)^{d_0/d_0-1} d\tau = \int a_{\rm E}^{d_0} d\tau\ . \tag{7.336}$$

Thus the corresponding metric of external space-time reads

$$g^{({\rm E})} = -a_E^{2d_0} d\tau \otimes d\tau + a_{\rm E}^2 g^{(0)}, \tag{7.337}$$

where for solutions (7.331)

$$a_{\rm E} = \left[\frac{(e^{qz^0})^{\frac{1}{q^2}}}{A_0 e^{\alpha^0\tau}}\right]^{\frac{1}{d_0-1}}, \tag{7.338}$$

and for solutions (7.333)

$$a_{\rm E} = \left(e^{pv^0}\right)^{\frac{1}{d_0-1}}. \tag{7.339}$$

Expressions for the internal scale factors are not affected. In Eqs. (7.337) to (7.339) the time τ is the harmonic one from the Brans-Dicke frame. The transformation to synchronous time in the Einstein frame is provided by Eq. (7.336). Once z^0 or v^0 is known as a function of τ, explicit expressions can be given. However these functions depends on the concrete form of the cosmological model (see [144],[119],[30],[138], [91],[121],[145]).

Above we obtained a general prescription for the generation of solutions in the Einstein frame from already known ones in the Brans-Dicke frame. It can easily be seen that the behavior of the solutions in both of these frames is quite different. Let us demonstrate this explicitly by the example of a generalized Kasner solution.

7.6.2 Solutions in original form

Let $t := t_{\rm BD}$ be the synchronous time of the Brans-Dicke frame, and $\dot{x}$ denote the derivative of x with respect to t.

The well-known Kasner solution [111] describes a 4-dimensional anisotropic space-time with the metric

$$g = -dt \otimes dt + \sum_{i=1}^{3} t^{2p_i} dx^i \otimes dx^i \tag{7.340}$$

where the p_i are constants satisfying

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} (p_i)^2 = 1. \tag{7.341}$$

It is clear that a multidimensional generalization of this solution is possible for a manifold with Ricci flat factor spaces (M_i, g_i), $i = 0, \ldots, n$. Particular solutions generalizing (7.340) with (7.341) were obtained in many papers [87],[146]-[150]. More general solutions for an arbitrary number of d_i-dimensional tori were found in [89] and generalized to the case of a free minimally coupled scalar field Φ in [83]. In the latter case there are two classes of solutions.

A first class represents namely Kasner-like solutions. None of these is contained in the hypersurface

$$\sum_{i=0}^{n} d_i \dot{\beta}^i = 0 \tag{7.342}$$

of constant spatial volume. With c and $a_{(0)i}$, $i = 0, \ldots, n$ integration constants, in the Brans-Dicke synchronous time gauge such a solution reads

$$a_i = a_{(0)i} t^{\overline{\alpha}^i}, \quad i = 0, \ldots, n, \tag{7.343}$$

$$\Phi = \ln t^{\overline{\alpha}^{n+1}} + c, \tag{7.344}$$

where the $\overline{\alpha}^i$ fulfill the conditions

$$\begin{aligned} \sum_{i=0}^{n} d_i \overline{\alpha}^i &= 1, \\ \sum_{i=0}^{n} d_i (\overline{\alpha}^i)^2 &= 1 - (\overline{\alpha}^{n+1})^2. \end{aligned} \tag{7.345}$$

Without an additional non-trivial scalar field Φ, i.e. for $\overline{\alpha}^{n+1} = 0$, these conditions become analogous to (7.341)

$$\sum_{i=0}^{n} d_i \overline{\alpha}^i = \sum_{i=0}^{n} d_i (\overline{\alpha}^i)^2 = 1. \tag{7.346}$$

Solution (7.343) describes a universe with increasing total spatial volume

$$v \sim \prod_{i=0}^{n} {a_i}^{d_i} \sim t \tag{7.347}$$

and decreasing Hubble parameter for each factor space

$$h_i := \frac{1}{a_i}\frac{da_i}{dt} = \frac{\overline{\alpha}^i}{t}, \quad i = 0, \ldots, n. \tag{7.348}$$

In the case of imaginary scalar field ($(\overline{\alpha}^{n+1})^2 < 0$) factor spaces with $\overline{\alpha}^i > 1$ undergo a power law inflation. The absence of a non-trivial scalar field, i.e. $\Phi \equiv 0$, implies (except for $d_0 = \overline{\alpha}^0 = 1$, $\overline{\alpha}^i = 0$, $i = 1, \ldots, n$) that $|\overline{\alpha}^i| < 1$ for $i = 0, \ldots, n$. In [151] it was shown that after a transformation $t \to t_0 - t$ (reversing the arrow of time) factor spaces with $\overline{\alpha}^i < 0$ can be interpreted as inflationary universes with scale factors $a_i \sim (t_0 - t)^{\overline{\alpha}^i}$ with $\overline{\alpha}^i < 0$ growing at an accelerated rate $\ddot{a}_i/a_i > 0$.

A second (more special) class of solutions is confined to the hyperplane (7.342) in momentum space. In this case (in the Brans-Dicke frame) harmonic and synchronous time coordinates coincide and solutions read

$$a_i = a_{(0)i}e^{b^i t}, \quad i = 0, \ldots, n, \tag{7.349}$$

$$\Phi = b^{n+1}t + c, \tag{7.350}$$

where c is a integration constant and the constants b^i satisfy

$$\sum_{i=0}^{n} d_i b^i = 0, \tag{7.351}$$

$$\sum_{i=0}^{n} d_i (b^i)^2 + (b^{n+1})^2 = 0.$$

The latter relation shows that these solutions are only possible if Φ is an imaginary scalar field with $(b^{n+1})^2 < 0$. For the sake of generality, such exact solutions for MCM are discussed here too for both frames, although as mentioned above in the Introduction they are unstable for classical theory. The main difference between dilatonic fields and Φ is that the dilatonic scalar fields of the MCM have a pure geometrical nature while Φ was from the very beginning just added by hand in the usual manner without any assumption on being real or imaginary. Therefore these exact solutions could be considered as unphysical from the point of classical stability.

The inflationary solution (7.349) describes a universe with constant total spatial volume

$$v \sim \prod_{i=0}^{n} a_i^{\,d_i} = \prod_{i=0}^{n} a_{(0)i}^{d_i}\ , \tag{7.352}$$

and a nonzero but constant Hubble parameter

$$h_i = \frac{1}{a_i}\frac{da_i}{dt} = b^i\ , \quad i = 0, \ldots, n\ , \tag{7.353}$$

for each factor space. This is a particular case of a steady state universe where stationarity of matter energy density in the whole universe is maintained due to redistribution of matter between contracting and expanding parts (factor spaces) of the universe (matter density in the whole universe is constant due to the constant volume). This is unlike the original steady-state theory [82], where a continuous creation of matter is required in order to stabilize matter density, which then necessitates a deviation from Einstein theory. In [32] the inflationary solution was generalized for the case of a σ-model with k-dimensional target vectors Φ rather than a single scalar field.

7.6.3 Solutions in the Einstein frame

In [25] several reasons have been listed why minimal coupling between geometry and matter and hence the Einstein frame is the preferred choice. There also a general prescription for the transformation of well known solutions from the Brans-Dicke frame to the Einstein frame has been given. It was demonstrated explicitly that qualitative cosmological features change significantly under this transformation. This was shown for a couple of examples, including the general multidimensional Kasner solution and a special inflationary solution with constant internal volume. In particular it was shown that inflationary solutions in the Brans-Dicke frame transform into non-inflationary ones in the Einstein frame. It is to be expected that this is a rather general feature, whence the multitude of solutions which appear inflationary in the Brans-Dicke frame will be indeed non-inflationary when considered in the Einstein frame.

Let us now transform the solutions (7.343) to (7.344) and (7.349) to (7.350) above to the Einstein frame.

We first consider the Kasner-like solution (7.343), where (6.39) determines the conformal factor as

$$\Omega^{-1} = \left(\prod_{i=1}^{n} e^{d_i \beta^i} \right)^{\frac{1}{D_0 - 2}} = C_1 t^{(1 - d_0 \overline{\alpha}^0)/(d_0 - 1)} \tag{7.354}$$

with

$$C_1 := \left(\prod_{i=1}^{n} a_{(0)i}^{d_i} \right)^{\frac{1}{D_0 - 2}} . \tag{7.355}$$

As it was noted above, the conformal transformation to the Einstein frame does not exist for $D_0 = 2$ ($d_0 = 1$). In the special case of $\overline{\alpha}^0 = \frac{1}{d_0}$ the conformal factor Ω is constant, and both frames represent the same connection, hence the same geometry. [1] Even in this case, (7.355) is still divergent for $d_0 = 1$.

The external space scale factor in the Einstein frame is defined by formula (7.326) which for (7.354) reads

$$a_{\rm E} = \Omega^{-1} a_{\rm BD} = \overline{a}_0 t^{(1 - \overline{\alpha}^0)/(d_0 - 1)}, \tag{7.356}$$

where $\overline{a}_0 := C_1 a_{(0)0}$. At $\overline{\alpha}^0 = \frac{1}{d_0}$ the (external space) scale factor $a_{\rm E} = \overline{a}_0 t^{\overline{\alpha}^0} \sim a_{\rm BD}$ has the same behavior in both frames which is just what one expects for constant Ω. The metric of the external space-time then reads

$$\overline{g}^{(\rm E)} = -\Omega^{-2} dt \otimes dt + a_{\rm E}^2 g^{(0)} = -dt_{\rm E} \otimes dt_{\rm E} + a_{\rm E}^2 g^{(0)}, \tag{7.357}$$

[1] Here is meant the geometry as given by the connection. Locally at $x \in \overline{M}_0$ this is just the $\mathrm{End}(T_x \overline{M}_0)$-valued Riemannian curvature 2-form.

where t is given synchronous time in the Brans-Dicke frame connected with synchronous time in the Einstein frame via (7.327). Putting the integration constant to zero we obtain

$$t = C_2 t_{\mathrm{E}}^{(d_0-1)/d_0(1-\overline{\alpha}^0)}, \tag{7.358}$$

where $C_2 = \left[C_1^{-1}\frac{1-\overline{\alpha}^0}{d_0-1}d_0\right]^{(d_0-1)/d_0(1-\overline{\alpha}^0)}$. The value $\overline{\alpha}^0 = 1$ is a singular one. It can be seen from (7.345) that $|\overline{\alpha}^i| < 1$, $i = 0, \ldots, n+1$ when the scalar field is real. The value $\overline{\alpha}^0 = 1$ may appear only in the case of an imaginary scalar field. (7.356) shows that, in this case a_{E} is a constant. In the case $\overline{\alpha}^0 \neq 1$ the generalized Kasner-like solutions in the Einstein frame take the form

$$\begin{aligned} a_{i,\mathrm{E}} &= \tilde{a}_i {t_{\mathrm{E}}}^{\tilde{\alpha}^i}, \quad i = 0, \ldots, n, & (7.359)\\ \Phi &= \tilde{\alpha}^{n+1} \ln t_{\mathrm{E}} + c. & (7.360) \end{aligned}$$

Here and in the following $a_{0,\mathrm{E}} := a_E(t_E)$, $a_{i,\mathrm{E}} := a_i(t_E)$, $i = 1, \ldots, n$, are given as functions of t_{E}, while $\tilde{a}_i$, $i = 0, \ldots, n$, and c are constants. In (7.359) and (7.360) the powers $\tilde{\alpha}^i$ are defined as

$$\begin{aligned} \tilde{\alpha}^0 &:= \frac{1}{d_0} & (7.361)\\ \tilde{\alpha}^i &:= \frac{d_0-1}{d_0(1-\overline{\alpha}^0)}\overline{\alpha}^i, \quad i = 1, \ldots, n+1, & \end{aligned}$$

with $\overline{\alpha}^i$, $i = 0, \ldots, n+1$, satisfying relations (7.345). Hence in contrast to (7.343) there is no freedom in the choice of the power $\tilde{\alpha}^0$. For example at $d_0 = 3$ one obtains an external space scale factor $a_{\mathrm{E}} = {t_{\mathrm{E}}}^{1/3}$, i.e. the external space (M_0, g_0) behaves like a Friedmann universe filled with ultra stiff matter (which is equivalent to a minimally coupled scalar field).

Let us emphasize here once more that in the present approach the physical theory is modeled as a D_0-dimensional effective action with the space-time metric (7.357) in the Einstein frame ($f = 0$). All internal spaces are displayed in the external space-time as scalar fields, leading to a D_0-dimensional self-gravitating σ-model with self-interaction.

Let us transform now the inflationary solution (7.349) to the Einstein frame. For this solution the conformal factor and the external space scale factor read

$$\begin{aligned} \Omega^{-1} &= C_1 \exp\left(-\frac{d_0 b^0}{d_0-1}t\right), & (7.362)\\ a_{\mathrm{E}} &= \overline{a}_0 \exp\left(-\frac{b^0}{d_0-1}t\right), & (7.363) \end{aligned}$$

where C_1 is defined by (7.355) and $\overline{a}_0 = C_1 a_{(0)0}$. Note that the conformal transformation (7.362) breaks down for $D_0 = 2$ ($d_0 = 1$). This happens even in the special case of $b^0 = 0$. For the latter, Ω is constant, whence the connection and its geometry represented by both frames are the same. Here, the external space is static in both of them.

For $b^0 \neq 0$, synchronous times in the Brans-Dicke and Einstein frames are related by

$$t = \frac{d_0-1}{d_0 b^0} \ln(C_2 {t_{\mathrm{E}}}^{-1}), \tag{7.364}$$

where (taking a relative minus sign in (7.327)) $C_2 = C_1 \frac{d_0-1}{d_0 b^0}$. Thus in the Einstein frame scale factors have power-law behavior

$$a_{i,\mathrm{E}} = \tilde{a}_i t_{\mathrm{E}}{}^{\tilde{\alpha}^i}, \quad i = 0, \ldots, n, \tag{7.365}$$

with

$$\begin{aligned} \tilde{\alpha}^{(0)} &:= \frac{1}{d_0} \\ \tilde{\alpha}^{(i)} &:= -\frac{d_0 - 1}{d_0}\frac{b^i}{b^0} \quad i = 1, \ldots, n. \end{aligned} \tag{7.366}$$

Similar as for the Kasner-like solution, the inflationary solution transformed to the Einstein frame has no freedom in choice of the power $\tilde{\alpha}^{(0)}$. The external space scale factor behaves as $a_{0,\mathrm{E}} \sim t_{\mathrm{E}}^{1/d_0}$ (compare also (7.359) and (7.361)). The scalar field reads

$$\Phi = \tilde{\alpha}^{n+1} \ln t_{\mathrm{E}} + c, \quad \tilde{\alpha}^{n+1} := -\frac{d_0 - 1}{d_0}\frac{b^{n+1}}{b^0}. \tag{7.367}$$

Using (7.351) we obtain the sum rules

$$\begin{aligned} \sum_{i=0}^{n} d_i \tilde{\alpha}^i &= d_0, \\ (\tilde{\alpha}^{n+1})^2 &= 2 - d_0 - \sum_{i=0}^{n} d_i (\tilde{\alpha}^i)^2 < 0, \end{aligned} \tag{7.368}$$

whence the scalar field is imaginary.

The main lesson we learned in this section is the following: The dynamical behavior of scale factors and scalar fields strongly depends on the choice of the frame. For example in the case of solutions originating from the Kasner and inflationary solutions the external space scale factor in the Einstein frame behaves as t_{E}^{1/d_0} (except for the cases $\overline{\alpha}^0 = 1$ and $b^0 = 0$ where a_{E} is a constant). In this case there is no inflation of the external space, neither exponential nor power law (with power larger than 1). However, an inversion of time $t_E \to t_0 - t_E$ yields a solution $a_E \sim (t_0 - t_E)^{1/d_0}$. Since both $\ddot{a}, \dot{a} \to -\infty$ this solution undergoes deflation, whence the flatness and horizon problems are solvable on the stage of external space contraction [143].

7.7 Multidimensional m-component cosmology

In this section let $N_i := \mathrm{dim} M_i$ for $i = 1, \ldots, n$ denote the dimension of the factor space M_i from the multidimensional manifold (7.117), and its corresponding scale factor here be labeled by x_i.

The action of the cosmological models considered here is

$$S = \frac{1}{2\kappa^2} \int_M d^D x \sqrt{|g|} R[g] + S_{\partial M} + S_{\mathrm{pf}}, \tag{7.369}$$

where $S_{\partial M}$ is a boundary term (just cancelling the boundary contribution of the Einstein action after dimensional reduction) and S_{pf} is the action of a multicomponent perfect fluid as a matter source. Multicomponent systems are often employed in 4-dimensional cosmology, and in many cases they are adequate types of matter for descibing some early epochs in the history of the universe [105]. In comoving coordinates the energy-momentum tensor of such a source reads

$$T_N^M = \sum_{s=1}^{m} T_N^{M(s)}, \tag{7.370}$$

$$(T_N^{M(s)}) = \mathrm{diag}\left(-\rho^{(s)}(t), \underbrace{p_1^{(s)}(t), \ldots, p_1^{(s)}(t)}_{N_1 times}, \ldots, \underbrace{p_n^{(s)}(t), \ldots, p_n^{(s)}(t)}_{N_n times}\right) \quad , \tag{7.371}$$

Furthermore, we suppose that the barotropic equation of state for the perfect fluid components is given by

$$p_i^{(s)}(t) = \left(1 - h_i^{(s)}\right) \rho^{(s)}(t), \quad s = 1, \ldots, m, \tag{7.372}$$

with constants $h_i^{(s)}$. Here pressures and matter constants may be different in different factor spaces.

The equation of motion $\bigtriangledown_M T_0^{M(s)} = 0$ for the perfect fluid component described by the tensor (7.371) reads

$$\dot{\rho}^{(s)} + \sum_{i=1}^{n} N_i \dot{x}^i \left(\rho^{(s)} + p_i^{(s)}\right) = 0. \tag{7.373}$$

Using the equations of state (7.372), via (7.373) integrals of motion may be obtained in form of constants

$$A^{(s)} := \rho^{(s)} \exp\left[2\gamma_0 - \sum_{i=1}^{n} N_i h_i^{(s)} x^i\right], \quad s = 1, \ldots, m, \tag{7.374}$$

In dimension D (with gravitational constant κ^2), the set of Einstein equations $R_N^M - R\delta_N^M/2 = \kappa^2 T_N^M$ can be written as $R_N^M = \kappa^2[T_N^M - T\delta_N^M/(D-2)]$. Furthermore, like the multidimensional geometry itself, these equations decomposes blockwise to $R_0^0 - R/2 = \kappa^2 T_0^0$ and $R_{n_i}^{m_i} = \kappa^2[T_{n_i}^{m_i} - T\delta_{n_i}^{m_i}/(D-2)]$. Using (7.119)-(7.372), we obtain

$$\frac{1}{2}\sum_{i,j=1}^{n} G_{ij}\dot{x}^i\dot{x}^j + V = 0, \tag{7.375}$$

$$\begin{aligned} \ddot{x}^i + \dot{x}^i(\dot{\gamma}_0 - \dot{\gamma}) \quad &= \quad -\kappa^2 \sum_{s=1}^{m} A^{(s)} \left(h_i^{(s)} - \frac{\sum_{k=1}^{n} N_k h_k^{(s)}}{D-2}\right) \\ &\times \quad \exp\left[\sum_{i=1}^{n} N_i h_i^{(s)} x^i - 2(\gamma - \gamma_0)\right]. \end{aligned} \tag{7.376}$$

Here,

$$G_{ij} = N_i \delta_{ij} - N_i N_j \tag{7.377}$$

are the components of the minisuperspace metric,

$$V = \kappa^2 \sum_{s=1}^{m} A^{(s)} \exp\left[\sum_{i=1}^{n} N_i h_i^{(s)} x^i - 2(\gamma - \gamma_0)\right]. \tag{7.378}$$

(7.374) is used to replace the densities $\rho^{(s)}$ in (7.375), (7.376) by expressions of the functions $x^i(t)$.

After the gauge fixing $\gamma = F(x^1, \ldots, x^n)$ the equations of motion (7.376) are the Euler-Lagrange equations obtained from the Lagrangian

$$L = \mathrm{e}^{\gamma_0 - \gamma} \left(\frac{1}{2} \sum_{i,j=1}^{n} G_{ij} \dot{x}^i \dot{x}^j - V \right) \tag{7.379}$$

and the zero-energy constraint (7.375) .

In order to be able to discuss the question of the physical (conformal) frame [26, 25] we perform also the dimensional reduction to a $1 + N_1$-dimensional system. After this reduction the action (7.369) reads

$$\begin{aligned} S = \frac{1}{2\kappa_0^2} \int_{M_0} d^{N_1+1}x \sqrt{|g^{(0)}|} \prod_{i=2}^{n} e^{N_i x^i(t)} \\ \times \left\{ R[g^{(0)}] - \sum_{i,j=2}^{n} G_{ij} \dot{x}^i \dot{x}^j \right\} + S_{\mathrm{pf}}, \end{aligned} \tag{7.380}$$

where

$$M_0 := R \times M_1, \quad g^{(0)} := -e^{2\gamma(t)} dt \otimes dt + e^{2x^1(t)} g^{(1)}, \tag{7.381}$$

$$\kappa_0^{-2} := \kappa^{-2} \prod_{i=2}^{n} \int_{M_i} d^{N_i}x \sqrt{|g^{(i)}|}. \tag{7.382}$$

Thus the reduced action directly invokes a Brans-Dicke like conformal frame, given by the metric $g^{(0)}$ on the extrinsic space-time manifold M_0.

As the physically relevant case, we will assume now $N_1 = 3$ (as for $M_1 := M_1^3$ in the example below). Then the non-minimal coupling between the 4-dimensional metric fundamental tensor in the Brans-Dicke frame ${}^{(4)}g^{(BD)} := g^{(0)}$ and the scalar fields x_i, $i = 2, \ldots, n$, can then be reinterpreted as a non-constant Newton factor G, while the metric (7.118) reads

$$g =^{(4)} g^{(BD)} + \sum_{i=2}^{n} e^{2x^i(t)} g^{(i)}. \tag{7.383}$$

A conformal transformation with a factor

$$\Omega^{-2} = \prod_{i=2}^{n} e^{N_i x^i} \tag{7.384}$$

then yields the 4-dimensional metric fundamental tensor ${}^{(4)}g^{(E)}$ of the Einstein frame,

$${}^{(4)}g^{(E)} \;\; = \;\; {}^{4}g^{(BD)} \Omega^{-2} \quad . \tag{7.385}$$

Below we will apply the transformation to the Einstein frame in a concrete example.

Now we introduce an n-dimensional real vector space $\mathbf{R^n}$. By $e_1, \dots e_n$ we denote the canonical basis in $\mathbf{R^n}$ ($e_1 = (1, 0, \dots, 0)$ etc.). Hereafter we use the following vectors:
1. the vector x with components being the solution of the equations of motion

$$x = x^1(t)e_1 + \ldots + x^n(t)e_n, \tag{7.386}$$

2. m vectors u_s, each of them for one component of the perfect fluid

$$u_s = \sum_{i=1}^{n} \left(h_i^{(s)} - \frac{\sum_{k=1}^{n} N_k h_k^{(s)}}{D-2} \right) e_i. \tag{7.387}$$

Let $< .,. >$ be a symmetrical bilinear form defined on $\mathbf{R^n}$ such that

$$< e_i, e_j >= G_{ij}. \tag{7.388}$$

The form is nondegenerate and the inverse matrix to (G_{ij}) has the components

$$G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2-D}. \tag{7.389}$$

The form $< .,. >$ endows the space $\mathbf{R^n}$ with a metric, the signature of which is $(-, +, ..., +)$ [152], [118]. G_{ij} is used to introduce the covariant components of vectors u_s

$$u_i^{(s)} = \sum_{i=1}^{n} G_{ij} u_{(s)}^j = N_i h_i^{(s)}. \tag{7.390}$$

For them the bilinear form reads

$$< u_s, u_r >= \sum_{i=1}^{n} h_i^{(s)} h_i^{(r)} N_i + \frac{1}{2-D} \left[\sum_{i=1}^{n} h_i^{(s)} N_i \right] \left[\sum_{j=1}^{n} h_j^{(r)} N_j \right]. \tag{7.391}$$

A vector $y \in \mathbf{R^n}$ is called time-like, space-like or isotropic, if $< y, y >$ is smaller, greater than or equal to zero, correspondingly. The vectors y and z are called orthogonal if $< y, z >= 0$.

Using the notation $< .,. >$ and the vectors (7.386)-(7.387), we may write the zero-energy constraint (7.375) and the Lagrangian (7.379) in the form

$$E = \frac{1}{2} < \dot{x}, \dot{x} > + \kappa^2 \mathrm{e}^{2(\gamma-\gamma_0)} \sum_{s=1}^{m} A^{(s)} \mathrm{e}^{< u_s, x >} = 0, \tag{7.392}$$

$$L = \frac{1}{2} \mathrm{e}^{\gamma_0-\gamma} < \dot{x}, \dot{x} > - \kappa^2 \mathrm{e}^{\gamma-\gamma_0} \sum_{s=1}^{m} A^{(s)} \mathrm{e}^{< u_s, x >}. \tag{7.393}$$

Here we take the harmonic time gauge, whence

$$\gamma(t) = \gamma_0 = \sum_{i=1}^{n} N_i x^i. \tag{7.394}$$

From the mathematical point of view the problem consist in solving the dynamical system, described by a Lagrangian of the general form

$$L = \frac{1}{2} < \dot{x}, \dot{x} > - \sum_{s=1}^{m} a^{(s)} \mathrm{e}^{<u_s, x>}, \tag{7.395}$$

where $x, u_s \in \mathbf{R^n}$. It should be noted that the kinetic term $< \dot{x}, \dot{x} >$ is not a positively definite bilinear form as it is usually the case in classical mechanics. Due to the pseudo-Euclidean signature $(-, +, ..., +)$ of the form $< ., . >$ such systems may be called pseudo-Euclidean Toda-like systems as the potential given in (6.11) defines a well known in classical mechanics Toda lattices [90], [153].

Note that, we have to integrate the equations of motion following from the Lagrangian (7.395) under the zero-energy constraint. Although an additional constant term $-a^{(0)}$ (with $u_0 \equiv 0 \in \mathbf{R^n}$) in the Lagrangian (7.395) does not change the equations of motion, it nevertheless shifts the energy constraint from zero to

$$E \equiv \frac{1}{2} < \dot{x}, \dot{x} > + \sum_{s=1}^{m} a^{(s)} \exp[< u_s, x >] = -a^{(0)} \equiv -\kappa^2 A^{(0)}. \tag{7.396}$$

In our cosmological model, with (7.372) and (7.387), such a term corresponds to a perfect fluid with $h_i^{(0)} = 0$ for all $i = 1, \ldots, n$. This is in fact just a Zeldovich (stiff) matter component, which can also be interpreted as a minimally coupled real scalar field. Taking into account the possible presence of Zeldovich matter, we have now to integrate the equations of motion for an arbitrary energy level E.

7.7.1 A_m Toda chain solution

We start from the Lagrangian (7.395) and the energy constraint (7.396) with

$$n \geq m + 1 \quad , \quad m \geq 2 \quad . \tag{7.397}$$

Vectors u_s are required to obey the relations

$$< u_s, u_s > \; = \; u^2 > 0 \quad , \quad s = 1, \ldots, m \quad , \tag{7.398}$$

$$< u_r, u_{r+1} > \; = \; -\frac{1}{2} u^2 \quad , \quad , r = 1, \ldots, m-1 \quad , \tag{7.399}$$

$$\text{all the remaining} \quad < u_r, u_s > \; = \; 0 \quad , \tag{7.400}$$

where u is an arbitrary non-zero real number. The relations (7.398)-(7.400) impose some restrictions on the constants $h_i^{(s)}$ in the barotropic equations of state (7.372), depending on the number $n \geq 2$ of factor spaces M_i and their dimensions N_i. Using (7.391), the restrictions from (7.398)-(7.400) may be evaluated explicitely.

In this case the vectors u_s are space-like, linearly independent, and can be interpreted as root vectors of the Lie algebra $A_m = sl(m + 1, \mathbf{C})$. The Cartan matrix (K_{rs}) (see e.g.

[154, 155]) then reads

$$(K_{rs}) = \left(\frac{2 < u_r, u_s >}{< u_r, u_r >}\right) = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix} \quad . \tag{7.401}$$

Now, we choose in $\mathbf{R^n}$ a basis $\{f_1, \dots, f_n\}$ with the following properties

$$f_{s+1} = \frac{2u_s}{< u_s, u_s >} \quad , \quad s = 1, \dots, m \quad , \tag{7.402}$$

$$< f_1, f_i > = \eta_{1i} \quad , \quad i = 1, \dots, n \quad , \tag{7.403}$$

$$< f_{s+1}, f_k > = 0 \ , \ < f_k, f_l >= \eta_{kl} \quad , \quad s = 1, \dots, m; \quad k, l = m+2, \dots, n \tag{7.404}$$

with

$$(\eta_{ij}) = diag(-1, +1, \dots, +1) \quad , \quad i, j = 1, \dots, n \quad . \tag{7.405}$$

This basis contains, besides vectors proportional to u_s, additional vectors $f_{m+2}, \dots, f_n$, iff $n > m+1$. By the decomposition

$$x(t) = \sum_{i=1}^{n} q^i(t) f_i \tag{7.406}$$

w.r.t. this basis, with relations (7.398) - (7.400), (7.402) - (7.404) the Lagrangian (7.395) takes the form

$$\begin{aligned} L \; = \; & \frac{1}{2}\left(-\left(\dot{q}^1\right)^2 + \frac{4}{u^2}\left[\sum_{s=2}^{m+1} (\dot{q}^s)^2 - \sum_{p=2}^{m} \dot{q}^p \dot{q}^{p+1}\right] + \sum_{k=m+2}^{n} \left(\dot{q}^k\right)^2 \right) \\ & -a^{(1)} e^{2q^2 - q^3} - \sum_{r=3}^{m} a^{(r-1)} e^{2q^r - q^{r-1} - q^{r+1}} - a^{(m)} e^{2q^{m+1} - q^m} \quad . \end{aligned} \tag{7.407}$$

The equations of motion for $q^1(t)$, $q^{m+2}(t)$, $\dots$, $q^n(t)$ read

$$\ddot{q}^1(t) = 0 \ , \ \ddot{q}^{m+2} = 0 \ , \ \dots, \ \ddot{q}^n(t) = 0 \ . \tag{7.408}$$

Then,

$$q^1(t) = a^1 t + b^1 \tag{7.409}$$

$$q^k(t) = a^k t + b^k \ , \ k = m+2, \dots, n. \tag{7.410}$$

The other equations of motion for $q^2(t), \dots, q^{m+1}(t)$ follow from the Lagrangian

$$L_E = \sum_{s=2}^{m+1} (\dot{q}^s)^2 - \sum_{p=2}^{m} \dot{q}^p \dot{q}^{p+1} - \frac{u^2}{2}\left[a^{(1)} e^{2q^2 - q^3} + \sum_{r=3}^{m} a^{(r-1)} e^{2q^r - q^{r-1} - q^{r+1}} + a^{(m)} e^{2q^{m+1} - q^m}\right] \ . \tag{7.411}$$

The linear transformation

$$q^{s+1} \longrightarrow q^s - \ln C^s \ , \ s = 1, \dots, m \ , \tag{7.412}$$

where the constants $C^1, \dots, C^m$ have to satisfy

$$\sum_{s=1}^{m} K_{rs} \ln C^s = \ln \frac{u^2 a^{(r)}}{2}, \ r = 1, \dots, m \ , \tag{7.413}$$

brings the Lagrangian into the form

$$L_{A_m} = \sum_{s=1}^{m} (\dot{q}^s)^2 - \sum_{r=1}^{m-1} \dot{q}^r \dot{q}^{r+1} - e^{2q^1 - q^2} - \sum_{p=2}^{m-1} e^{2q^p - q^{p-1} - q^{p+1}} - e^{2q^m - q^{m-1}} \ . \tag{7.414}$$

The latter represents the Lagrangian of a Toda chain associated with the Lie algebra A_m [90] when the root vectors are put into the Chevalley basis and coordinates describing the motion of the mass center are separated out.

We use the method suggested in [156] for solving the equations of motion following from (7.414) and obtain

$$e^{-q^s} \equiv F_s(t) = \sum_{r_1 < \dots < r_s}^{m+1} v_{r_1} \cdots v_{r_s} \Delta^2(r_1, \dots, r_s) e^{(w_{r_1} + \dots + w_{r_s})t} \tag{7.415}$$

where $\Delta^2(r_1, \dots, r_s)$ denotes the square of the Vandermonde determinant

$$\Delta^2(r_1, \dots, r_s) = \prod_{r_i < r_j} \left(w_{r_i} - w_{r_j} \right)^2 \ . \tag{7.416}$$

The constants v_r and w_r have to satisfy the relations

$$\prod_{r=1}^{m+1} v_r = \Delta^{-2}(1, \dots, m+1) \ , \tag{7.417}$$

$$\sum_{r=1}^{m+1} w_r = 0 \ . \tag{7.418}$$

The energy of the Toda chain described by this solution is given by

$$E_0 = \frac{1}{2} \sum_{r=1}^{m+1} w_r^2 \ . \tag{7.419}$$

Finally, we obtain the folowing decomposition of the vector $x(t)$

$$x(t) = (a^1 t + b^1) f_1 + \sum_{s=1}^{m} \frac{-2 \left(\ln F_s(t) + \ln C^s \right)}{< u_s, u_s >} u_s + \sum_{k=m+2}^{m} (a^k t + b^k) f_k \ . \tag{7.420}$$

We remind the reader that the coordinates $x^i(t)$ of the vector $x(t)$ are, with respect to the *canonical* basis in $\mathbf{R^n}$, the logarithms of the scale factors in the corresponding cosmological model.

Let us introduce the vectors

$$\alpha = a^1 f_1 + \sum_{k=m+2}^{n} a^k f_k \equiv \sum_{i=1}^{n} \alpha^i e_i \in \mathbf{R}^n \ , \tag{7.421}$$

$$\beta = b^1 f_1 + \sum_{k=m+2}^{n} b^k f_k \equiv \sum_{i=1}^{n} \beta^i e_i \in \mathbf{R}^n \ , \tag{7.422}$$

with α^i, β^i being their coordinates with respect to the canonical basis. Using (7.403) and (7.404), we conclude these coordinates have to satisfy the following equations

$$< \alpha, u_s >= \sum_{i,j=1}^{n} G_{ij} \alpha^i u^j_{(s)} = 0 \ , \ s = 1, \dots, m \ , \tag{7.423}$$

$$< \beta, u_s >= \sum_{i,j=1}^{n} G_{ij} \beta^i u^j_{(s)} = 0 \ , \ s = 1, \dots, m \ , \tag{7.424}$$

where the $u^i_{(s)}$ are the coordinates of u_s in the canonical basis (see (7.387)).

The total energy E of the system is given by

$$E = \frac{1}{2} < \alpha, \alpha > + \frac{2}{u^2} E_0 = \frac{1}{2} \sum_{i,j=1}^{n} G_{ij} \alpha^i \alpha^j + \frac{1}{u^2} \sum_{s=1}^{m+1} (w_s)^2 \ . \tag{7.425}$$

If $n = m + 1$, then $< \alpha, \alpha >= - (a^1)^2 \leq 0$. With (7.425), we then obtain $E \leq \frac{2}{u^2} E_0$.

Finally, the scale factors of the multidimensional cosmological model with the Lagrangian (7.395) and the energy constraint (7.396) are given by

$$e^{x^i(t)} = \prod_{s=1}^{m} \left[\tilde{F}_s^2(t) \right]^{-u^i_{(s)}/< u_s, u_s >} e^{\alpha^i t + \beta^i} \quad , \tag{7.426}$$

where

$$\tilde{F}_s(t) = C^s \cdot F_s(t) \ , \ s = 1, \dots, m \ . \tag{7.427}$$

Using (7.374) we obtain the following solution for the densities of the perfect fluid components

$$\rho^{(1)} = A^{(1)} \mathrm{e}^{-2\gamma_0} \frac{\tilde{F}_2}{\tilde{F}_1^2} \ , \quad \rho^{(m)} = A^{(m)} \mathrm{e}^{-2\gamma_0} \frac{\tilde{F}_{m-1}}{\tilde{F}_m^2}$$

$$\rho^{(p)} = A^{(p)} \mathrm{e}^{-2\gamma_0} \frac{\tilde{F}_{p-1} \tilde{F}_{p+1}}{\tilde{F}_p^2} \ , \quad p = 2, \dots, m-1. \tag{7.428}$$

where γ_0 is defined by (7.121) and may be calculated by (7.426).

The constants C^s are specified by (7.413). The solution contains the parameters α^i, β^i, v_r, w_r ($i = 1, \dots, n$, $r = 1, \dots, m{+}1$) obeying the constraints (7.423),(7.424),(7.417), (7.418),(7.425). If the energy E is arbitrary (see (7.396)) the solution has $2n$ free parameters as required.

7.7.2 Example in Einstein frame

We consider a space-time manifold M

$$M = R \times M_1^3 \times M_2^3 \times M_3^4 \tag{7.429}$$

where M_1^3, M_2^3, and M_3^4 are factor spaces of dimension $N_1 = 3$, $N_2 = 3$, and $N_3 = 4$, respectively. The first component of the perfect fluid shall have the $h_i^{(1)}$ values

$$h_1^{(1)} = 0 \quad , \quad h_2^{(1)} = h \quad , \quad h_3^{(1)} = 0 \tag{7.430}$$

while the second component is given by

$$h_1^{(2)} = h \quad , \quad h_2^{(2)} = 0 \quad , \quad h_3^{(2)} = 0 \quad . \tag{7.431}$$

h is a real valued parameter with the restriction

$$h \neq 0. \tag{7.432}$$

Here, with $m = 2$, relations (7.398), (7.399) are fulfilled indeed.

In this case, the exact solution of the field equations gives the metric in the Brans-Dicke frame (for some remarks on the Einstein frame see below) by

$$\begin{aligned} g &= \left[\tilde{F}_1(t)\tilde{F}_2(t)\right]^{\frac{2}{h}} \\ &\times \Big\{ - e^{8\alpha_0 t + 8\beta_0} dt^2 + \left[\tilde{F}_2(t)\tilde{F}_1^4(t)\right]^{\frac{-2}{3h}} ds_1^2 + \left[\tilde{F}_1(t)\tilde{F}_2^4(t)\right]^{\frac{-2}{3h}} ds_2^2 + \frac{e^{2\alpha_0 t + 2\beta_0}}{\left[\tilde{F}_1(t)\tilde{F}_2(t)\right]^{\frac{2}{h}}} ds_3^2 \Big\} \end{aligned} \tag{7.433}$$

where $\alpha_0 \equiv \alpha^3$ and $\beta_0 \equiv \beta^3$ are integration constants. Furthermore here

$$\tilde{F}_1(t) = \kappa^2 \left[A^{(1)}\right]^{\frac{2}{3}} \left[A^{(2)}\right]^{\frac{1}{3}} h^2 F_1(t) \quad , \tag{7.434}$$

$$\tilde{F}_2(t) = \kappa^2 \left[A^{(1)}\right]^{\frac{1}{3}} \left[A^{(2)}\right]^{\frac{2}{3}} h^2 F_2(t) \quad , \tag{7.435}$$

$$F_1(t) = v_1 e^{w_1 t} + v_2 e^{w_2 t} + v_3 e^{w_3 t} \quad , \tag{7.436}$$

$$\begin{aligned} F_2(t) &= v_1 v_2 (w_1 - w_2)^2 e^{(w_1 + w_2)t} + v_1 v_3 (w_1 - w_3)^2 e^{(w_1 + w_3)t} \\ &+ v_2 v_3 (w_2 - w_3)^2 e^{(w_2 + w_3)t} \quad . \end{aligned} \tag{7.437}$$

In our case, the energy E is given by

$$E = -6\alpha_0^2 + \frac{1}{2h^2}\left[w_1^2 + w_2^2 + w_3^2\right] = -\kappa^2 A^{(0)} \quad . \tag{7.438}$$

$A^{(0)} > 0$ means that Zeldovich matter is present in all the internal spaces (See the remarks at the end of sect. 3). With $A^{(0)} = 0$ (7.438) is the energy constraint specialized to our example.

The nine parameters $w_1, w_2, w_3, v_1, v_2, v_3, \alpha_0, \beta_0, E$ have to satisfy (7.437) and the two further relations

$$w_1 + w_2 + w_3 = 0 \tag{7.439}$$

and

$$v_1v_2v_3 = [(w_1-w_2)(w_2-w_3)(w_1-w_3)]^{-2} \tag{7.440}$$

(See (7.418) and (7.417)!).

Finally, we have to give the expressions for the matter densities $\rho^{(1)}$ and $\rho^{(2)}$. They read

$$\rho^{(1)} = A^{(1)}\left[\tilde{F}_1^{-2-\frac{2}{h}}(t)\tilde{F}_2^{1-\frac{2}{h}}(t)\right]e^{-8\alpha_0 t-8\beta_0} \quad , \tag{7.441}$$

$$\rho^{(2)} = A^{(2)}\left[\tilde{F}_2^{-2-\frac{2}{h}}(t)\tilde{F}_1^{1-\frac{2}{h}}(t)\right]e^{-8\alpha_0 t-8\beta_0} \tag{7.442}$$

and their quotient is

$$\frac{\rho^{(2)}}{\rho^{(1)}} = \frac{A^{(2)}}{A^{(1)}}\left(\frac{\tilde{F}_1(t)}{\tilde{F}_2(t)}\right)^3 \quad . \tag{7.443}$$

The solution is invariant under $(w_1, w_2, v_1, v_2) \to (w_2, w_1, v_2, v_1)$. Still there is a lot of freedom for a solution. Hence it is difficult to identify general properties of the solutions. What one can say is the following: We know that

$$\left(e^{x^1(t)}\right)^{-3h} \propto \left|\frac{F_1(t)}{F_2^2(t)}\right| \tag{7.444}$$

and

$$\left(e^{x^2(t)}\right)^{-3h} \propto \left|\frac{F_2(t)}{F_1^2(t)}\right| \quad . \tag{7.445}$$

An easy but tedious discussion of the different possibilities of choosing the parameters w_1 and w_2 shows that the expressions (7.444) and (7.445) have for $t \to \pm\infty$ the following shape:

$$\left|\frac{F_2(t)}{F_1^2(t)}\right| \quad \overset{t\to\pm\infty}{\longrightarrow} \quad e^{f_{\pm\infty}(w_1,w_2)t} \tag{7.446}$$

and

$$\left|\frac{F_1(t)}{F_2^2(t)}\right| \quad \overset{t\to\pm\infty}{\longrightarrow} \quad e^{g_{\pm\infty}(w_1,w_2)t} \tag{7.447}$$

with some functions $f_{\pm\infty}$, and $g_{\pm\infty}$ of the parameters w_1 and w_2 being negative for $+\infty$ and positive for $-\infty$. This shows that the scale factors of the manifold M_1 and M_2 go to infinity for $t \to \pm\infty$. As for the scale factor $e^{x^3(t)}$ of the manifold M_3, we have

$$\left(e^{x^3(t)}\right)^{3h} \propto e^{3\alpha_0 ht} \tag{7.448}$$

where $k(w_1, w_2)$ is some positive function of the parameters w_1 and w_2.

The proper time T as a function of harmonic time t is given by integration of $dT = e^{\gamma_0}dt$ with $\gamma_0 = 3x^1 + 3x^2 + 4x^3$. For $t \to \pm\infty$ the behavior of both, the proper time T and the scale factor e^{x^3} depends on the choice of α_0. This holds for the case that in all manifolds Zeldovich matter is present. If the case is excluded then α_0 is given by the energy constraint.

Let us now also consider the metric in the Einstein frame for the example considered here. The conformal factor (7.384)

$$\Omega^{-2} = e^{3x^2+4x^3} \tag{7.449}$$

and the metric for the Einstein frame is given in the gauge (7.394) by

$$g^{(E)} = e^{3x^2+4x^3}\left(-e^{2(3x^1+3x^2+4x^3)}dt\otimes dt + e^{2x^1}g^{(1)}\right) + \sum_{i=2}^{3} e^{2x^i}g^{(i)} \quad . \tag{7.450}$$

Finally,

$$T_E = \int e^{3x^1+\frac{9}{2}x^2+6x^3}dt \tag{7.451}$$

is the cosmic (i.e. proper, i.e eigen) time in the Einstein frame.

Although we have specialized already to an example, there are still too many parameters for a complete analytic discussion of the solution in this metric frame with T being the eigen time coordinate. To give an idea, we consider the following case:

$$w_3 = 0 \quad , \tag{7.452}$$

$$w_2 = -w_1 < 0 \quad , \tag{7.453}$$

$$v_1 = v_2 = \frac{1}{2w_1^2} = \frac{v_3}{2} \quad , \tag{7.454}$$

$$\alpha_0 = \frac{w_1}{\sqrt{6}h} \quad . \tag{7.455}$$

Eq. (7.455) results from the requirement that the factor spaces here do not share a common additional Zeldovich matter contribution. In the following, the constants $A^{(1)}$ and $A^{(2)}$, not essential for a discussion of the solution's qualitative behavior, are chosen to simplify expressions. Then,

$$F_1(t) = F_2(t) = \frac{1}{w_1^2}\left[1+\frac{1}{2}\left(e^{w_1 t}+e^{-w_1 t}\right)\right] \quad . \tag{7.456}$$

F_1 is a *positive* and time symmetric function. Therefore all scale factors are strictly positive.

With all these requirements, after some constant rescalings $g^{(E)} \mapsto ds^2$, $g^{(i)} \mapsto ds_i^2$, the metric reads

$$ds^2 = -F_1^{5/h}e^{12\alpha_0 t}dt\otimes dt + F_1^{5/3h}e^{4\alpha_0 t}g^{(1)} + F_1^{2/3h}g^{(2)} + e^{2\alpha_0 t}g^{(3)} \quad . \tag{7.457}$$

The proper time T_E is a solution of

$$dT_E \propto F_1^{5/2h}e^{6\alpha_0 t}dt \quad . \tag{7.458}$$

For $t \to +\infty$, the choice (7.455) yields

$$T_E \propto e^{\left(\frac{5}{2}+\sqrt{6}\right)\frac{w_1 t}{h}} \quad . \tag{7.459}$$

Needless to say that the choices of parameters above have mainly been made for the sake of mathematical simplicity. The study of more realistic models, though beyond the scope of the present paper, could be a topic for future investigations. Our example above just served to demonstrate the general method.

7.8 2-dimensional dilaton gravity

In the case $D_0 = 2$, the gauge $f \stackrel{!}{=} 0$ does not exist, and the corresponding conformal transformation, which for $D_0 \neq 2$ connects the gauge $\gamma \stackrel{!}{=} 0$ with the gauge $f \stackrel{!}{=} 0$, is singular for $D_0 = 2$ (cf. [26]). Let us adopt therefore the gauge $\gamma \stackrel{!}{=} 0$, and set

$$
\begin{aligned}
\sigma &:= -\frac{1}{2} {}^{(s)}q z^1 = -\frac{1}{2} \ln(\phi) \ , \\
m &:= \frac{1}{{}^{(s)}q^2} = \frac{D'-1}{D'} \ , \\
k &:= -\frac{D'}{D'+1} \ .
\end{aligned}
\tag{7.460}
$$

Then, the action takes the string-like form

$$
{}^{(s)}S = \frac{1}{2\kappa_0^2} \int_{M_0} \mathrm{d}^2 x \sqrt{|g^{(0)}|} \quad e^{-2\sigma} \left\{ R[g^{(0)}] + 4m(\partial\sigma)(\partial\sigma) - \sum_{i=2}^{n} (\partial z^i)(\partial z^i) \right. \\
\left. e^{-2(\frac{1}{k}+m)\sigma} + \sum_{i=1}^{n} R[g^{(i)}] e^{-2\sum_{j=2}^{n} (T^{-1})^i_j z^j} \right\} \ , \tag{7.461}
$$

where T^{-1} is the inverse of the homogeneous linear transformation (6.30).

The conformal transformation $g^{(0)} \mapsto \tilde{g}^{(0)} := e^{-2m\sigma} g^{(0)}$ then yields

$$
{}^{(s)}S = \frac{1}{2\kappa_0^2} \int_{M_0} \mathrm{d}^2 x \sqrt{|\tilde{g}^{(0)}|} \quad e^{-2\sigma} \left\{ R[\tilde{g}^{(0)}] - \sum_{i=2}^{n} \tilde{g}^{(0)\lambda\nu} \frac{\partial z^i}{\partial x^\lambda} \frac{\partial z^i}{\partial x^\nu} \right. \\
\left. + e^{-\frac{2}{k}\sigma} \sum_{i=1}^{n} R[g^{(i)}] e^{-2\sum_{j=2}^{n} (T^{-1})^i_j z^j} \right\} \ , \tag{7.462}
$$

where the dilatonic field σ has become kinetically irrelevant. This peculiarity of the case $D_0 = 2$ has been discussed in [26] in more detail. In [29] a particular application is given by a $2+3$-dimensional model with spherical symmetry.

7.8.1 Reduction of inhomogeneous cosmology to dimension 2

Let us now consider in more details the dimensional reduction to a space-time of dimension $D_0 = 2$. In this case the conformal transformation to the Einstein frame is singular, whence the model can not be expressed in this frame. This is not a fault of the theory, but rather corresponds to the well known fact that 2-dimensional Einstein equations are empty, i.e. they do not imply a dynamics [157, 158]. Thus we shall consider 2-dimensional dilaton gravity only.

We start with the case with one dilaton, $n = 1$. The action can be written in the 'string-like' form [159, 160, 161]

$$
S = \frac{1}{2\kappa_0^2} \int_{M_0} \mathrm{d}^2 x \sqrt{|\det g^{(0)}|} e^{-2\sigma} \left\{ R[g^{(0)}] + 4m g^{(0)\lambda\nu} \frac{\partial\sigma}{\partial x^\lambda} \frac{\partial\sigma}{\partial x^\nu} - 2\Lambda e^{-2(\frac{1}{k}+m)\sigma} \right\}, \tag{7.463}
$$

where

$$\begin{aligned} \sigma &:= -\frac{1}{2} d_1 \beta^1 \ , \\ m &:= \frac{d_1 - 1}{d_1} \ , \\ k &:= -\frac{d_1}{d_1 + 1} \ , \\ 2\Lambda &:= -R[g^{(1)}] \ . \end{aligned} \tag{7.464}$$

By a conformal transformation of $g^{(0)}_{\mu\nu}$ to

$$\tilde{g}^{(0)}_{\mu\nu} = e^{-2m\sigma} g^{(0)}_{\mu\nu} \ , \tag{7.465}$$

we can formulate the action without kinetic dilation term, as

$$S = \frac{1}{2\kappa_0^2} \int_{M_0} \mathrm{d}^2x \sqrt{|\det \tilde{g}^{(0)}|} e^{-2\sigma} \left\{ \tilde{R}[\tilde{g}^{(0)}] - 2\Lambda e^{-\frac{2}{k}\sigma} \right\} . \tag{7.466}$$

The 2d actions (7.463) and (7.466) are invariant under homogeneous conformal transformations

$$\begin{aligned} \check{g}^{(0)}_{\mu\nu} &:= \Omega^{-2} \tilde{g}^{(0)}_{\mu\nu} \ , \\ \check{g}^{(1)}_{\mu\nu} &:= \Omega^{-2} g^{(1)}_{\mu\nu} \ , \end{aligned} \tag{7.467}$$

where Ω is constant. Applying (7.467) with

$$\Omega^2 := -\frac{d_1}{(d_1+1)^{1+\frac{1}{d_1}}} \frac{1}{2\Lambda}$$

yields

$$2\check{\Lambda} := -\check{R}[\check{g}^{(1)}] = -\frac{d_1}{(d_1+1)^{1+\frac{1}{d_1}}} = \frac{k}{(k+1)^{1+\frac{1}{k}}} \tag{7.468}$$

and the action (7.466) now reads

$$S = \frac{1}{2\kappa_0^2} \int_{M_0} \mathrm{d}^2x \sqrt{|\det \check{g}^{(0)}|} e^{-2\sigma} \left\{ \check{R}[\check{g}^{(0)}] - 2\check{\Lambda} e^{-\frac{2}{k}\sigma} \right\} . \tag{7.469}$$

If we assume that the dilaton field is specifically given through the geometry on M_0 and the dimension d_1 of M_1, according to

$$e^{-2\sigma} := (k+1) \left(\check{R}[\check{g}^{(0)}] \right)^k , \tag{7.470}$$

then the action (7.469) takes the form [162, 160, 161, 163]

$$\begin{aligned} S &= \frac{1}{2\kappa_0^2} \int_{M_0} \mathrm{d}^2x \sqrt{|\det \check{g}^{(0)}|} \left(\check{R}[\check{g}^{(0)}] \right)^{k+1} \\ &= \frac{1}{2\kappa_0^2} \int_{M_0} \mathrm{d}^2x \sqrt{|\det \check{g}^{(0)}|} \left(\check{R}[\check{g}^{(0)}] \right)^{\frac{1}{d_1+1}} . \end{aligned} \tag{7.471}$$

In the general case of multi-scalar fields, the kinetic term of the dilaton can be removed by an obvious analogous procedure. The 'string-like' form of the action then is

$$S = \frac{1}{2\kappa_0^2}\int_{M_0} \mathrm{d}^2x\sqrt{|\det g^{(0)}|}e^{-2\sigma}\left\{R[g^{(0)}] + 4mg^{(0)\lambda\nu}\frac{\partial\sigma}{\partial x^\lambda}\frac{\partial\sigma}{\partial x^\nu} - \sum_{i=2}^{n} g^{(0)\lambda\nu}\frac{\partial z^i}{\partial x^\lambda}\frac{\partial z^i}{\partial x^\nu} \right.$$
$$\left. - e^{-2(\frac{1}{k}+m)\sigma}\sum_{i=1}^{n} 2\Lambda_i e^{-2\sum_{j=2}^{n} U^i_j z^j}\right\}, \quad (7.472)$$

where now

$$\begin{aligned}
\sigma &:= -\frac{1}{2}qz^1 , \\
m &:= \frac{1}{q^2} = \frac{D'-1}{D'} , \\
k &:= -\frac{D'}{D'+1} , \\
2\Lambda_i &:= -R[g^{(i)}].
\end{aligned} \quad (7.473)$$

With (7.473), the conformal transformation (7.465) yields

$$S = \frac{1}{2\kappa_0^2}\int_{M_0} \mathrm{d}^2x\sqrt{|\det \tilde{g}^{(0)}|}e^{-2\sigma}\left\{\tilde{R}[\tilde{g}^{(0)}] - \sum_{i=2}^{n} \tilde{g}^{(0)\lambda\nu}\frac{\partial z^i}{\partial x^\lambda}\frac{\partial z^i}{\partial x^\nu} \right.$$
$$\left. - e^{-\frac{2}{k}\sigma}\sum_{i=1}^{n} 2\Lambda_i e^{-2\sum_{j=2}^{n} U^i_j z^j}\right\}. \quad (7.474)$$

In (7.474) there is no kinetic term of the dilaton field. The kinetic terms of all extra scalar fields z^i have the normal sign. The extra fields z^i play the role of usual matter, coupling to the dilaton field σ.

7.8.2 Example: dilaton gravity from 5-dimensional Einstein gravity

We start from the 5-dimensional gravitation action

$$S = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^5x\sqrt{|g|}R[g] + S_{\text{GHY}} \quad (7.475)$$

with the Gibbons-Hawking-York boundary term S_{GHY}, and apply the formalism of dimension reduction from last section. $\mathcal{M}$ is a manifold of dimension $D = 5$, factorizing into a a Lorentzian manifold $\mathcal{M}_0$ of dimension $D_0 = 1+1$ and a compact homogeneous Riemannian manifold $\mathcal{M}_1$ of dimension $d_1 = 3$,

$$\mathcal{M} = \mathcal{M}_0 \times \mathcal{M}_1 \ . \quad (7.476)$$

We may choose conformal light cone coordinates x^+ and x^- on $\mathcal{M}_0$ such that the metric is

$$g^{(0)} = -e^{2\rho}dx^+ \otimes dx^- , \quad (7.477)$$

whence in these coordinates $g^{(0)}_{11} \equiv g_{++} = 0$, $g^{(0)}_{12} \equiv g_{+-} = -\frac{1}{2}e^{2\rho}$, and $g^{(0)}_{22} \equiv g_{--} = 0$ while the metric on $\mathcal{S}^3$ is homogeneous. On $\mathcal{M}$ the metric is defined by the warped metric

$$g = g^{(0)} + e^{2\beta} g^{(1)} \tag{7.478}$$

where $\beta \equiv -\phi$ is a warp function depending on x^+ and x^-. This corresponds to $n = 1$ and a gauge $\gamma = 0$. The effective 2-dimensional action here is

$$^{(s)}S = \frac{1}{4\kappa_0^2} \int_{\mathcal{M}_0} dx^- dx^+ \sqrt{|g^{(0)}|} e^{3\beta} \left\{ R\left[g^{(0)}\right] + 6 g^{(0)\,\lambda\nu} \frac{\partial\beta}{\partial x^\lambda} \frac{\partial\beta}{\partial x^\nu} + R\left[g^{(1)}\right] e^{-2\beta} \right\} \tag{7.479}$$

where λ and ν take the values 1 or 2.

Another equivalent formulation is reached after introduction of the Brans-Dicke (BD) field, for $n = 1$ defined via

$$\Phi \equiv e^{-d_1\phi} := e^{d_1\beta} \ . \tag{7.480}$$

Then, the action (7.479) reads

$$^{(s)}S = \frac{1}{4\kappa_0^2} \int_{\mathcal{M}_0} dx^- dx^+ \sqrt{|g^{(0)}|} \left\{ \Phi R\left[g^{(0)}\right] + \frac{2}{3} g^{(0)\,\lambda\nu} \frac{1}{\Phi} \frac{\partial\Phi}{\partial x^\lambda} \frac{\partial\Phi}{\partial x^\nu} + R\left[g^{(1)}\right] \Phi^{\frac{1}{3}} \right\} \ . \tag{7.481}$$

With $n = 1$, the negative Brans-Dicke parameter is $-\omega = 1 - (1/d_1) = \frac{2}{3}$, and, with $d_1 = 3$, the last exponent is $1 - (2/d_1) = \frac{1}{3}$. Note that according to (7.480), a BD field $e^{-3\phi}$ is typical for a reduction of our $(2+3)$-dimensional theory, while that of a $(2+2)$-dimensional theory, like e.g. in [164], would be $e^{-2\phi}$. With

$$\begin{aligned} \sigma &:= -\frac{3}{2}\beta \quad , \\ m &:= \frac{2}{3} \quad , \\ k &:= -\frac{3}{4} \quad , \\ \Lambda &:= -3 \quad , \end{aligned} \tag{7.482}$$

the action (7.481) takes the 'string-like' form

$$^{(s)}S = \frac{1}{4\kappa_0^2} \int_{\mathcal{M}_0} dx^- dx^+ \sqrt{|g^{(0)}|} e^{-2\sigma} \left\{ R\left[g^{(0)}\right] + \frac{8}{3} g^{(0)\,\lambda\nu} \frac{\partial\sigma}{\partial x^\lambda} \frac{\partial\sigma}{\partial x^\nu} + R\left[g^{(1)}\right] e^{\frac{4}{3}\sigma} \right\} \ . \tag{7.483}$$

After a conformal transformation

$$g^{(0)}_{\lambda\nu} \mapsto \tilde{g}^{(0)}_{\lambda\nu} := e^{-\frac{4}{3}\sigma} g^{(0)}_{\lambda\nu} \ , \tag{7.484}$$

the action is free of any kinetic dilaton term. Explicitly, it reads

$$^{(s)}S = \frac{1}{4\kappa_0^2} \int_{\mathcal{M}_0} dx^- dx^+ \sqrt{|\tilde{g}^{(0)}|} e^{-2\sigma} \left\{ \tilde{R}\left[\tilde{g}^{(0)}\right] + R\left[g^{(1)}\right] e^{\frac{8}{3}\sigma} \right\} \ . \tag{7.485}$$

Furthermore, the action is invariant with respect to constant conformal transformations

$$\begin{aligned} \check{g}^{(0)}_{\lambda\nu} &:= \Omega^{-2} \tilde{g}^{(0)}_{\lambda\nu} \quad , \\ \check{g}^{(1)}_{\lambda\nu} &:= \Omega^{-2} \tilde{g}^{(1)}_{\lambda\nu} \ . \end{aligned} \tag{7.486}$$

Let us choose now $\Omega^2 := 4^{1/3}8$. If we assume the geometries on $\mathcal{M}_1$ and $\mathcal{M}_0$ to be related by

$$e^{-2\sigma} \stackrel{!}{=} \frac{1}{4}\left(\check{R}\left[\check{g}^{(0)}\right]\right)^{-3/4} , \tag{7.487}$$

the action (7.485) takes the higher order-form

$${}^{(s)}S = \frac{1}{4\kappa_0^2}\int_{\mathcal{M}_0} dx^- dx^+ \sqrt{|\check{g}^{(0)}|}\left(\check{R}\left[\check{g}^{(0)}\right]\right)^{1/4} . \tag{7.488}$$

7.8.3 Spherically symmetric model coupled to matter

We now restrict to field configurations with 3-spherical symmetry,

$$\mathcal{M} = \mathcal{M}_0 \times \mathcal{S}^3 , \tag{7.489}$$

where, with $g^{(1)} = d\Omega_3^2$, the 5-dimensional metric can be written as

$${}^{(5)}ds^2 = {}^{(2)}ds^2 + e^{-2\phi}d\Omega_3^2 , \qquad {}^{(2)}ds^2 \equiv g^{(0)} = -e^{2\rho}dx^+ \otimes dx^- , \tag{7.490}$$

and the curvature scalar of the unit 3-sphere is $R\left[g^{(1)}\right] = 6$. As in the previous section, we use a conformal parametrization $g_{+-} = -\frac{1}{2}e^{2\rho}$, $g_{++} = g_{--} = 0$, in light cone coordinates $x^\pm = (x^0 \pm x^1)$ on M_0. ϕ is a dilatonic field corresponding to M_0-dependent, but homogeneous, dilations of the 3-sphere $\mathcal{S}^3$. The solutions of the classical action are, of course, the known 5-dimensional Schwarzschild solutions which are periodic in imaginary time and hence have the usual temperature for the 5-dimensional case. The metric (7.490) then yields the Ricci curvature scalar

$${}^{(5)}R = 6e^{2\phi} + e^{-2\rho}(48\partial_+\phi\partial_-\phi - 24\partial_+\partial_-\phi) + {}^{(2)}R , \qquad {}^{(2)}R = 8e^{-2\rho}\partial_+\partial_-\rho , \tag{7.491}$$

and the 2-dimensional dilaton-gravity action of (7.481), with (7.480), specializes to

$$\begin{aligned} {}^{(2)}S &= \frac{1}{4\kappa_0^2}\int_{\mathcal{M}_0} dx^- dx^+ \sqrt{|g^{(0)}|}e^{-3\phi}({}^{(2)}R + 6(\nabla\phi)^2 + 6e^{2\phi}) \\ &= \frac{1}{2\kappa_0^2}\int_{\mathcal{M}_0} dx^+ dx^- e^{-3\phi}(-2\partial_+\partial_-\rho + 6\partial_+\phi\partial_-\phi - \frac{3}{2}e^{2\phi+2\rho}) , \end{aligned} \tag{7.492}$$

which is analogous to the classical actions of the CGHS model [159] and the model of Lowe [164] (in these models the 2-dimensional coupling constant is gauged to $\kappa_0^2 \equiv \frac{\pi}{2}$), except for the coefficient of the kinetic dilatonic term and, more importantly, the presence of a dilatonic factor $e^{-3\phi}$ (the characteristic BD field for a model reduced from a $2+3$-dimensional one) instead of the usual $e^{-2\phi}$ (the typical BD field for an effective reduction of a $2+2$-dimensional model). Therefore, when ρ is constant, in (7.492) the last term can no longer be interpreted as a cosmological constant, as for its analogue according to [159, 164].

Since the divergence term of the Liouville field ρ in (7.492) is the same as in [159, 164], the semiclassical quantum corrections, induced by coupling ρ to N additional matter fields in 2 dimensions and integrating out their fluctuations, yield the same terms for (7.492) as in [159, 164]. Hence, in the conformal gauge (7.490), we obtain the effective action

$$S_{\rm eff} = \frac{1}{2\kappa_0^2}\int_{\mathcal{M}_0} dx^+dx^- \Big[e^{-3\phi}(-2\partial_+\partial_-\rho + 6\partial_+\phi\partial_-\phi) \\ -\frac{3}{2}e^{-\phi+2\rho} - \frac{1}{2}\sum_{j=1}^{N}\partial_+ f_j\partial_- f_j + \frac{N}{12}\partial_+\rho\partial_-\rho\Big] \ , \tag{7.493}$$

where f_j, $j = 1, \dots, N$ are massless non-dilatonic scalar matter fields. The equation of motion derived from (7.493) with dilatonic scalar field ϕ, Liouville field ρ, and matter fields f_j, are

$$\partial_+\partial_-\rho = 2\partial_+\partial_-\phi - 3\partial_+\phi\partial_-\phi - \frac{1}{4}e^{2\phi+2\rho} \tag{7.494}$$

$$\partial_+\partial_-\phi = \frac{3}{2}\partial_+\phi\partial_-\phi + \frac{1}{8}e^{2\phi+2\rho} + \frac{\frac{1}{2}}{1-\frac{N}{36}e^{3\phi}}(\partial_+\partial_-\phi + \frac{1}{4}e^{2\phi+2\rho}) \tag{7.495}$$

$$\partial_+\partial_- f = 0. \tag{7.496}$$

To these equations of motion we have to add the constraints which arise from the energy-momentum tensor of (7.493) associated to the vanishing metric components g_{++} and g_{--} in conformal gauge. We then obtain Ricci curvature tensor components

$$R_{\pm\pm}[g] = 3\partial_\pm^2\phi - 6\partial_\pm\rho\partial_\pm\phi - 3(\partial_\pm\phi)^2 \ . \tag{7.497}$$

Hence, the additional energy-momentum constraints from (7.493) read

$$T_{\pm\pm} := e^{-3\phi}[6\partial_\pm\rho\partial_\pm\phi - 3\partial_\pm^2\phi + 3(\partial_\pm\phi)^2] \\ +\frac{1}{2}(\partial_\pm f)^2 - \frac{N}{12}[(\partial_\pm\rho)^2 - \partial_\pm^2\rho + t_\pm] \overset{!}{=} 0, \tag{7.498}$$

where $t_\pm \equiv t_\pm(x_\pm)$ are functions of integration to be fixed by the boundary conditions.

In the equations above, quantum corrections are represented, according to [159, 164], by the term proportional to the number N of (non-dilatonic scalar) matter fields. These corrections become small as one recedes from the (black hole) singularity. The curvature singularity occurs at the point where equation (7.495) degenerates, i.e. at a critical value of the dilaton field

$$\phi = \phi_{\rm cr} := -\frac{1}{3}\ln\frac{N}{18} \ , \tag{7.499}$$

which is different (namely larger resp. smaller for $N > \frac{128}{3}$ resp. $N < \frac{128}{3}$) than the critical value $-(1/2)\ln(N/24)$ where the singularity appears for the black-hole models obtained from a 2-dimensional reduction of $2+2$-dimensional Einstein gravity (see [164]).

7.8.4 Static solutions and the horizon problem

We shall consider now static solutions, of two different classes: First, finite-mass solutions, and secondly, infinite-mass solutions which represent black holes supported by an incoming flux of radiation.

According to [164], the most natural choice for the radial coordinate corresponding to the finite-mass static solutions is

$$\sigma := \frac{1}{2}(x^+ - x^-) \equiv x^1 \ , \tag{7.500}$$

i.e. the radial coordinate is nothing but just the space like coordinate on the world sheet of the string. It is in this sense that the finite-mass static solutions correspond to a four-dimensional space-time having the structure of a wormhole. In terms of the coordinate σ, the equations of motion (7.494) and (7.495) reduce to

$$\begin{aligned} \phi'' &= \frac{3}{2}(\phi')^2 - \frac{1}{2}e^{2\phi+2\rho} + \frac{1}{2}\frac{\phi'' - e^{2\phi+2\rho}}{1 - \frac{N}{36}e^{3\phi}} && (7.501) \\ \rho'' &= 2\phi'' - 3(\phi')^2 + e^{2\phi+2\rho}, && (7.502) \end{aligned}$$

where $(\cdot)' := \frac{d}{d\sigma}(\cdot)$. With $f_j = 0$, the two constraint equations from (7.498) reduce in the static case to

$$e^{-3\phi}[\frac{3}{2}\rho'\phi' - \frac{3}{4}\phi'' + \frac{3}{4}(\phi')^2] - \frac{N}{48}[(\rho')^2 - \rho'' + t] \stackrel{!}{=} 0. \tag{7.503}$$

Here $t \equiv t(\sigma)$ is a σ-dependent function of integration which has to be chosen consistently with the static boundary conditions. We now solve (7.501)-(7.503) in two limiting situations.

First, we note that the vacuum corresponds to solutions with $t = 0$. When obtained from the equations above, such a vacuum coincides with that found by Lowe in the 2D model derived by dimensional reduction of the 4-dimensional Einstein action [164], i.e.

$$\phi = -\ln\sigma \ , \quad \rho = 0. \tag{7.504}$$

We turn now to the asymptotic ($\sigma \to \infty$) solution of equations (7.501)-(7.503). These will be obtained by solving the linearized equations of motion about vacuum (7.504) for $t = 0$. The resulting asymptotic finite-mass solution is

$$\phi = -\ln\sigma + \frac{2M}{\sigma^2}, \quad \rho = \frac{M}{\sigma^2}, \tag{7.505}$$

which, in turn, shows a different behaviour with respect to that of the corresponding solutions obtained by Lowe in his 4-dimensional example [164]. Equations (7.501) and (7.503) are different from the corresponding equations of [164]. Especially, with (7.505), terms containing N do not cancel. This means that with growing N the value of σ must increase too.

With $\tau := (1/2)(x^+ + x^-) \equiv x^0$ and $\sigma := (1/2)(x^+ - x^-) \equiv x^1$, in linear approximation, solution (7.505) corresponds to a 2-dimensional metric

$${}^{(2)}ds^2 = \left(1 + \frac{M}{\sigma^2}\right)^2 (-d\tau^2 + d\sigma^2), \tag{7.506}$$

which describes two asymptotically Euclidean regions connected by a throat of a size proportional to M. This is typical for a Euclidean Tolman-Hawking wormhole with ${}^{(5)}R = 0$, as one should expect [165]. This striking result supports the idea that there is a close connection between wormholes and the formation and evaporation of black holes [166, 167].

According to [164], the most natural choice of the radial coordinate for the class of static solutions having regular event horizons is $r^2 := -x^+x^-$, with the horizons at $r = 0$. Setting now $(\cdot)' := \frac{d}{dr}(\cdot)$, the equations of motion become

$$\phi'' + \frac{\phi'}{r} = \frac{3}{2}\phi'^2 - \frac{1}{2}e^{2\phi+2\rho} + \frac{1}{2}\frac{\phi'' + \frac{\phi'}{r} - e^{2\phi+2\rho}}{1 - \frac{N}{36}e^{3\phi}} , \tag{7.507}$$

$$\rho'' + \frac{\rho'}{r} = 2\phi'' + \frac{2\phi'}{r} - 3\phi'^2 + e^{2\phi+2\rho} , \tag{7.508}$$

and, for $f_j = 0$, the constraint equations read

$$e^{-3\phi}[6\phi'\rho' - 3\phi'' + \frac{3\phi'}{r} + 3\phi'^2] \stackrel{!}{=} \frac{N}{12}[\rho'^2 - \rho'' + \frac{\rho'}{r} + \frac{\tilde{t}}{r^2}] , \tag{7.509}$$

where $\tilde{t} \equiv \tilde{t}(r)$ is a r-dependent function of integration consistent with regularity conditions at the event horizon.

Regularity of (7.507) to (7.509) at the event horizon ($r = 0$) may be maintained with non-trivial boundary conditions

$$\phi_h \stackrel{!}{=} \phi_{\rm cr} , \tag{7.510}$$

$$\phi'_h \stackrel{!}{=} \frac{1}{2}\rho'_h , \quad \tilde{t}_h \stackrel{!}{=} 0 , \tag{7.511}$$

with $\phi_{\rm cr}$ from (7.499). These boundary conditions by themselves do not directly determine precise values for $\rho'_h = 2\phi'_h$ and ρ_h at $r = 0$, whence the general solution can be parametrized in terms of ρ_h and ρ'_h.

The solution satisfying the special boundary condition $\rho'_h = \rho_h = 0$ is converging in the limit $r \to \infty$, to the vacuum (7.504), for $\phi_h < \phi_{\rm cr}$, and to the vacuum solutions of (7.507) and (7.508) for $\phi_h = \phi_{\rm cr}$. The latter takes the form

$$e^{-\phi} = a - b\ln r , \quad \rho = \ln\frac{b}{r} , \tag{7.512}$$

with constants a and b. This is indeed a solution of the same structure as that obtained in [164] for the 2 + 2-dimensional case. So, as $\phi_h \to \phi_{\rm cr}$, the solution approaches a limiting form which cannot be at zero temperature, which suggests then the possibility that there exists a naked singularity in this limit. This can be seen by approximating the evaporation of a large mass black hole by a succession of these static solutions and computing the outgoing energy flux

$$\epsilon = \frac{N}{12}[\partial_-^2\rho - (\partial_-\rho)^2] = \frac{N}{48b_0^2} ,$$

which gives a finite non zero value, even at the limiting form of the solutions, as far as the parameter b_0 for that limiting form is finite [164].

However, if we choose boundary conditions with (7.510), then a black hole cannot be formed until the dilaton field reaches exactly the value at which the curvature singularity appears as dressed by an emerging horizon. If we allow a large black hole to exist with variable parameter ϕ_h, then, as this parameter approaches $\phi_{\rm cr}$, the 2-dimensional geometry, and the transformation required to trivialize the values of ρ and ρ' on the horizon, can no longer be determined uniquely, whence ρ_h and ρ_h' might take any value. Although $\phi_h = \phi_{\rm cr}$ is typically true for some solution of a particular asymptotic form, it might be no longer true in a range of masses that would be appropriate for discussing evaporation [168].

Moreover, it is known from detailed numerical studies that one cannot make any firm conclusions by looking only at the static solutions (see e.g. [164]). In the present case, one would expect initially $\phi_h > \phi_{\rm cr}$ and, as evaporation proceeds, ϕ_h approaching $\phi_{\rm cr}$. Whether or not for all solutions this evolution would stop at $\phi_h = \phi_{\rm cr}$ can be decided only by more detailed investigations of the corresponding solutions. One possibility would be to trace the horizon and the singularity numerically as was done in [164].

Recently, several classes of rather general dilaton-gravity models with integrable potentials have been identified in [169]. In fact the dimensionally reduced dilaton-gravity model (7.492) is integrable. However, the effective model (7.493), which includes additionally N matter fields and the induced quantum corrections from their coupling to the 2-geometry via the Liouville field ρ, does not directly belong to any of the integrable classes treated by [169]. However, at present we do not know, whether (7.493) can be reformulated as an integrable model, whence one might be able to find the exact solution for the receding horizon problem explicitly.

Because of the particular difficulties one faces in the model (7.493) with both, exact integration and numerical analysis, a better understanding of its solutions, and the receding horizon problem in particular, will only be achieved after further investigations, which at present have to be postponed to possible future work.

8. Discussion

Above three different structures on manifolds and geometries are discussed which admit applications in three different quite directions of mathematical cosmology.

The part on causal structures and their application in quantum general relativity is based on most recent work. Accordingly, this part more than others still needs maturation and further development. However, the metric-independent definition of a causal structure via topologically defined local cones opens an interesting possibility to discuss a generalized Haag-Kastler system of axioms of quantum (field) theory on differential manifolds without a fixed metric background. This is particularly interesting for a general relativistic approach to QFT and as an ad hoc possibility to quantize geometry with fixed causal structure as as a topological background.

Of course there then still remains the question, how strong a causal background structure can be in order to leave still sufficiently many degrees of freedom for quantum geometry. How much more freedom a given causal structure admits for the remaining geometry as compared to a conformal structure ?

Above in Sect. 2 we presented topological definitions of local (i.e. pointwise) cone (LC) structures for a general topological or differentiable manifold M of dimension $d+1>2$ and notions of causality on M in a purely topological manner. It is remarkable that such definitions are possible, whence the usual recursion to a Lorentzian metric becomes redundant.

Proposition 1 gives criteria which *locally* distinguish the exterior and the interior of the cone at any point from each other. Proposition 3 and Example 1 provide concrete global topological conditions for M in order to allow the relative distinction of interior and exterior of all its cones. Minkowski space is obviously a manifold which satisfies the conditions for topologically distinguished interior and exterior according to Example 1. It is however a priori not clear what for each given category CAT of manifolds is the largest class of manifolds with the topological structure described in Example 1.

We saw that a global consistent distinction between future and past cones requires just a topological $\mathbf{Z}_2$-connection. Note that, as an important possible application, the canonical approach to quantum gravity comes always along with such a connection. In fact the canonical configuration variables for oriented manifolds may be there be chosen as $\mathrm{SO}(d+1)$-connections.

The presented LC structures, C-causality, and other our purely topological causality notions provide a possibility to define causality in quantum theories of quantum gravity. In contrast to other approaches based on a much weaker local notion of causality on *sets*, which essentially involve only a partial ordering, the present definition gives the possibility to work with a local definition of causality on differentiable *manifolds* which still captures the essential notions for curves in a C-causal manifold to be lightlike, timelike or spacelike without the need of an underlying Lorentzian structure. For any set S in a C-causal manifold a topological notion of a causal complement $S^{\perp}$ is given. Any double cone $\mathscr{K}$ in a C-causal manifold has the duality property $\mathscr{K}^{\perp\perp}=\mathscr{K}$.

For a more general background independent quantum theory the restriction of local diffeomorphisms to those consistent with a strong LC structure on the whole manifold might appear too restrictive. After all, a strong LC structure implies already the existence of a conformal metric, whence diffeomorphisms may be restricted locally to conformal ones. Nevertheless note that even a strong LC structure is much more flexible than a conformal metric structure. The local cones of different vertices might refold away from their vertices with rather complicated intersection topologies while a CAT continuous conformal metric within its (regular!) domain does not admit refolding singularities of the characteristic surfaces, each of the which is spanned out by all the null geodesics passing through a given vertex. Of course refolding and the associated singularities should be a topic of further more systematic investigations elsewhere.

A strong LC-structure on all of M already implies the existence of a conformal metric structure and a requirement of compatibility with that metric would reduce the local covariance group to local conformal diffeomorphisms. One might however also weaken the LC and causal structure of the manifold by considering in any leaf Σ of a given foliation only cones with vertices on $\Gamma\subset\Sigma$ instead on all of Σ. A natural choice for Γ is the dual graph of a triangulation. Then, the cones have to (CAT-)vary along the edges, but at least for CAT$\supset\mathcal{C}^\infty$ the cones at the vertices of the graph can be freely ascribed. Consequently, a geometry constructed on that basis will be invariant under diffeomorphisms much more general than conformal ones.

Let us however also emphasize that, although the existence of a local conformal metric is guaranteed by a strong LC structure, it is a priori not obvious that this metric should play any significant rôle. However, similarly, also the need to restrict diffeomorphisms to those compatible with the conformal metric may be questioned. One might eventually expect that within some approach to quantum geometry a cone at a vertex $p \in O \subset \Sigma$ should be replaced by an appropriate average over cones with vertices within some region O of minimal Heisenberg uncertainty. Then the flexibility of the weak and strong LC structures makes them interesting concepts and potential ingredients for a possible definition of quantum causality too.

Classically, the existence of a local metric requires only the differentiable structure in an arbitrary small neighborhood of the vertex, and the defined LC structures fix the preferred null directions only locally at each vertex. With sufficiently strong notions of causality (e.g. C-causality above) the null structures of this metric may become consistent with the global structure of cones of the LC structure. Note that in the case of a given Lorentz metric null geodesics lie on cones, and with sufficiently strong causality, e.g. global hyperbolicity, these cones have to be consistent with respect to each other and under variation of the vertex without refolding into each other (i.e. in particular without conjugate points).

For Lorentzian manifolds there is a hierarchy of common notions of causality which have been generalized above. Provided our definitions of causality are sufficiently natural it should be possible to prove (at least parts) of this hierarchy in the more general topological setting. However a complete investigation of the mutual relations between different topological causality concepts is beyond the scope and goal of the present paper.

It should be emphasized that the above was just brief demonstration of the possibility to introduce notions of cones and causality on CAT topological manifolds without a metric. In particular, weak and strong LC structures, C-causality, precausality, and some generalizations of the most common notions of causality have been obtained. However the investigation is far from complete. It remains for future work to develop the topological approach to causal structures on manifolds further, to investigate better some of its implications, and last not least to demonstrate its advantages in background independent formulations of algebraic quantum field theory and quantum gravity.

Sect. 3 gives a first idea of possible applications of the causal structures from the previous Sect. 2 in quantum general relativity.

Sect. 3.1 presented an axiomatic introduction to algebraic QFT on manifolds, generalizing the elegant Haag-Kastler formulation of quantum field theory on Minkowski space. Sect. 3.1.1 introduced the general axioms of QFT on a differentiable manifold (including in particular isotony and covariance) which do not require a particular notion of causality, and Sect. 3.1.2 suggests further axioms for QFT on a manifold with cone causality, all in closest possible analogy to usual AQFT.

Sect. 3.2 showed how quantum geometry may be discussed as an algebraic quantum theory within this axiomatic framework. In [4] it was shown how quantum geometry may fit into the framework of algebraic QFT. This setting naturally accommodates spin networks on graphs. This was used in [9] to investigate the classical limit of quantum geometry. If quantum gravity is a true AQFT with infinitely many vertices, the classical limit yields a tubular network of resolutions of vertices and edges. Remarkably, resolutions like these had already previously been discussed in [10]) from a different point of view. This results obtained should be motivation for further investigations, e.g. on the classical limit of the geometrical operators.

The work of the second part describes some progress in understanding homogeneous structures, in particular for (pseudo-)Riemannian manifolds.

Results of Sect. 4 and 5 are a continuation of previous work from [19] and [20]. The relationship between the classifying spaces of local homogeneous 3-geometries and their isometries is now understood more deeply.

The classifying space of local isometries in fixed real dimension is given as non-separating T0-space of corresponding Lie-algebras. In [19] such classifying spaces had already been constructed up to real dimension 4.

The structure of K^3 is understood now in relation to the manifold spanned within the classifying space of constant isometry of homogeneous 3-geometries. K^3 takes the shape of Morse-like isometry potential, that indicates the level of metastability. The latter can be achieved by rigidity of the isometry for a geometry corresponding to an interior point in that manifold.

It was shown that the topology of such a classifying space can be understood as the dual of the Zariski topology in terms of the Lie algebra cohomology related to deformation within the category of Lie algebras.

The construction of classifying spaces of local homogeneous Riemannian 3-geometries is based on scalar geometric invariants as described in [14]. For the Riemannian case, in contrast to our earlier approaches where we only evaluated the three scalar invariants from the Ricci tensor, here we additionally evaluate a scalar invariant constructed from the first covariant derivative of the Ricci tensor. Coincidence of these four invariants implies local isometry in the set of homogeneous Riemannian 3-manifolds, whereas the three eigenvalues of the Ricci tensor do not suffice to distinguish the local homogeneous 3-geometries.

This isometry classification can be used to study homogeneous deformations of 3-dimensional factor spaces of classical cosmological models. The complete parametrization of local homogeneous 3-geometries is of particular interest for a systematic approach to their canonical quantization. The spatially homogeneous class is of primary importance for quantum cosmology.

The analogous scalar invariants have been evaluated also in 3 cases of Lorentzian signature with the help of computer algebra. As a partial result, one obtains qualitative differences between each of these cases. One of the main problems with the Lorentzian signature is to obtain a systematic control of the various possible relative orientations of the light cone axis with respect to the principal axes of anisotropy in tangent space. The relationship between the triad of the tangent frame and the signature, have to be examined in more detail, and also more generally than here, such that e.g. also null eigen directions of the triad can be taken into account systematically.

As an application of an isometry classification in the Lorentzian case, one can study e.g. the rigity of minisuperspace isometries of multidimensional geometries [16].

These new, though still partial, results obtained in the second part on homogeneous structures indicate an interesting direction for future research.

The large third part on multidimensional geometries deals with the effective sigma-model [23] for an Einstein-Hilbert action for multidimensional geometries. Various applications and solutions in multidimensional cosmology derive from this model. It provided for the first time a systematic geometrical description of bosonic string theory sectors on a curved rather than background. In [26] the first such description has been obtained for a multidimensional space-time background. Solutions with p-branes have been obtained first in [31].

Sect. 6 analyzed the mathematical structure of the multidimensional σ-model, in particular in 6.1 the model with pure gravitational action from pure multidimensional geometry, and in 6.2 the model extended to scalars and $p+2$ forms. The orthobrane condition (7.15) allowed us to find exact solutions. According to the second Theorem in 6.3 it is a sufficient condition for the target space of the σ-model to be a locally symmetric space. It was shown, that apart from cases with degenerate coupling matrix (6.58), the orthobrane case is the generic one where the target space is locally symmetric.

The structure of the multidimensional sigma-model should admit the application of covariant quantization techniques in arbitrary effective dimension. For certain classical solutions with scalars, covariant quantization schemes could be applied rather directly. The existence and applicability of reformulations of the sigma-model as a matrix-model or BF-theory is also under present investigation.

The form (6.41) suggests that, in the effective dimension $3+1$, the sigma-model in the Einstein frame should admit a canonical quantization with self-dual connections, similar as for Einstein gravity with minimally coupled scalar matter fields.

Furthermore, the extension of the sigma-model to Riemann-Hilbert manifolds is currently under investigation.

In 6.4 convenient coordinate gauges on the base space were presented, which were used in particular for the solutions of the model presented in Sect. 7. In 7.1 solutions with Abelian (flat) target space were presented. In 7.2 orthobrane solutions with flat potential ${}^{(E)}V = 0$ in Einstein gauge were obtained. In 7.3, examples of a certain minimal static, spherically symmetric p-brane configuration are given with just one electric and one magnetic antisymmetric F component (since in 4 dimensions we only deal with a single electromagnetic field), which in general intersect and interact with a single scalar field. Spherical symmetry here is considered in the physical relevant $D_0 = 4$ case of S^2 spheres, although the extension to arbitrary spheres is straightforward.

According to the final Theorem of 6.3, besides popular families of orthobrane solutions there are further families of solutions, which have another additional symmetry, e.g. coinciding F-field charges for the electro-magnetic solutions described in 7.4.2. In the target space this additional symmetry is expressed by a linear dependency (7.99) between column vectors Y of the coupling matrix L defined in (6.58).

For the static solutions of 7.4, Hawking temperature T_H can be formally calculated by surface gravity via a Komar-like integral. For both, the orthobrane case and the case of equal charges $Q_e^2 = Q_m^2$, the expressions of T_H depend characteristically on the intersection dimension. This results are also interesting in the context of recent increased interest in extremal p-brane configurations with black holes [170].

The interpretation of the extremal limit $k \to 0$ is delicate. The solutions above have been described in isotropic coordinates which cover just the asymptotically flat exterior of the black hole. A better understanding of their global causality structure would require an investigation of the maximal extension of the space-time rather than only of its exterior part. The limit $k \to 0$ is called extremal, since via (7.94) and (7.113) in this limit the effective asymptotic mass M is just given by the charges, $G_N M = \frac{1}{2}(Bp_{\rm e} + Cp_{\rm m})$ and $G_N M = p/(1 + d_2)$, respectively. Further work is required to understand this type of extremality, and the related asymptotics of T_H. In the limit $k \to 0$ the latter depends critically on the intersection dimension $d_2 + 1$ of the electric and magnetic brane. If $d_2 = 0$ and both charges are nonzero, this temperature tends to zero in this limit; if $d_2 = 1$ and both charges are nonzero, it tends to a finite limit, and in all other cases it tends to infinity. T_H remains finite for $d_2 = 1$ and becomes infinite for $d_2 \geq 2$. As it was pointed out recently in [171] particular care is needed in order to associate the correct physical charges and thermal properties of a black hole correctly with its horizon.

Global properties of the p-brane black hole solutions like causal structure, boundary conditions, black hole thermodynamics and extremality are issues under current investigation.

Finally, structural analogies to investigations in [77] suggest that it should be possible to apply solution generating techniques like the Ehlers-Harrison transformation also in the context of the multidimensional σ-model.

In Sect. 7.5 we considered the generalization of a homogeneous cosmological model of Bianchi type I to an anisotropic multidimensional one with $n \geq 2$ Ricci-flat spaces of arbitrary dimensions, in the presence of m homogeneous non-interacting minimally coupled scalar fields. Under certain conditions these models are equivalent to multidimensional cosmological models in the presence of an m-component perfect fluid with equations of state $P^{(a)} = \left(\alpha^{(a)} - 1\right) \rho^{(a)}$ with matter constants $\alpha^{(a)}$ for $a = 1, \ldots, m$. Using this equivalence, for $m = 3$, we find integrable models when one of the scalar fields is equivalent to an ultra-stiff perfect fluid component, the second one corresponds to dust, and the third one is equivalent to a vacuum component. Recent investigations [172], [173] suggest to apply this 3-component model for an explanation of the current phase of accelerated expansion.

The dynamics of the universe was investigated in general, as well as in a particular 3-component integrable case. For integrable models, there are four qualitatively different types of evolution of the universe, depending on the potential $U(z_0)$, but in all four cases the universe has a Kasner-like behaviour near the cosmological singularity.

In the cases where the universe can expand to infinity, an isotropization takes place and results in an asymptotically de Sitter universe.

In quantum cosmology, instantons, solutions of the classical Einstein equations in Euclidean space, play an important role, giving significant contributions to the path integral. They are connected with the changing geometry of the model. We found here three interesting types of instantons. The first one describes tunnelling between a Kasner-like universe and an asymptotically de Sitter universe. Sewing a number of these instantons may provide the Coleman mechanism for the vanishing of the cosmological constant. Another type of instanton is responsible for the birth of the universe from "nothing". It was shown that corresponding Lorentzian solutions can ensure inflation of the external space and compactification of the internal ones. This problem deserves a more detailed investigation in future research. The third type of instantons describes the Euclidean space which has an asymptotically anti-de Sitter wormhole geometry.

The scalar field potentials $U^{(a)}(\varphi^{(a)})$ $(a = 1, \ldots, m)$ can be reconstructed in general. We performed this procedure for integrable models, and exact forms of potentials.

The equivalence between a scalar field and a perfect fluid component helps also to investigate the quantum behaviour of the universe. We obtained the Wheeler-de Witt equation from the effective perfect fluid Lagrangian. Exact solutions are found, some of which describe cosmological transitions with a signature change of the metric. e.g. universe nucleation as quantum tunnelling from an Euclidean region. Other solutions are given as quantum wormholes with discrete spectrum.

In Sect. 7.6 the general transformation of solution into the Einstein frame is given. Known solutions in the Brans-Dicke frame are transformed into this frame and reinterpreted (again in cosmic proper time).

Typical solutions for considered models in the Brans-Dicke frame have a general structure described either by (7.331) or (7.333). For solutions of this type the transformation to Einstein frame is given by (7.338) and (7.339) respectively. The qualitative difference induced by the distinct functions z^0 and v^0 respectively necessitates a separate treatment of these two classes. In any case, solutions to a given model in the Einstein frame show a quite different dynamical behaviour from the corresponding solutions in the Brans-Dicke frame when they are compared in the physically relevant cosmic synchronous time of the respective frame. (Although the conformal time is indeed the same in both frames, here it is not the physically relevant one.)

We demonstrated this explicitly on the example of the generalized Kasner solution (7.343) (and exceptional inflationary solutions (7.349)). With respect to the proper time in Einstein frame, the external space scale factor $a_{0,\mathrm{E}}$ has a surprisingly simple and definite root law behavior $a_{0,\mathrm{E}} \sim t_{\mathrm{E}}^{1/d_0}$ (except for the case of an exotic imaginary scalar field where $a_{0,\mathrm{E}}$ may be constant). Hence this model does not admit inflation of the external space in Einstein frame although it can undergo deflation.

In Sect. 7.7 a multidimensional cosmological model with m-component perfect fluid source and independent pressures in n Ricci-flat factor spaces $(n-1 \geq m \geq 2)$ is characterized by certain vectors, related to the matter constants in the barotropic equations of state for fluid components of all factor spaces. We showed that, in the case where we can interpret these vectors as the root vectors of a Lie algebra of Cartan type $A_m = sl(m+1, \mathbf{C})$, the model reduces to the classical open m-body Toda chain. Using an elegant technique by Anderson [156] for solving this system, we integrated the Einstein equations for the model and presented the metric in a Kasner-like form.

Finally, Sect. 7.8 described a 2-dimensional dilaton-gravity model obtained via dimensional reduction from a higher dimensional Einsteinian gravity model. The effective model obtained from a $2+3$-dimensional Einstein-Hilbert action was investigated in the particular case of spherical internal 3-geometry. Static black holes solutions have been used to discuss qualitatively the receding horizon problem. It turned out that a black hole cannot be formed until the dilaton field reaches exactly the value at which the curvature singularity appears as dressed by an emerging horizon.

To summarize, although the present work analyzes only some particular aspects of causal, homogeneous, and multidimensional structures on manifolds and geometries arising in contemporary mathematical investigations of gravitation and cosmology, it clearly demonstrated the richness and variety of mathematical problems involved. Therefore it is very likely that "Mathematical Cosmology" is going to establish as a growing field of future research by its own.

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Thesen zur Habilitationsschrift

1. *Zum Thema der Arbeit*
Die vorliegende Arbeit verdeutlicht den erweiterten Umfang, den die mathematische Kosmologie als Teilgebiet der mathematischen Physik aufgrund des aktuellen Stands der Forschung inzwischen eingenommen hat. Insbesondere die Verallgemeinerung von der klassischen $1+3$-dimensionalen Raumzeit auf beliebige multidimensionale Geometrien, sowie die Untersuchung topologisch-relativistischer kausale Strukturen in Hinblick auf eine quantisierte Geometrie der Raumzeit, wurden als relativ junge Erweiterungen der mathematischen Kosmologie insbesondere auch in [1] vertreten.
Der erste Teil, über kausale Strukturen auf Mannigfaltigkeiten und ihre Anwendung in der modernen mathematischen Kosmologie, basiert auf den relativ jungen Arbeiten [2], [3], [4], [5], [6], [7], und [8]. Die Anwendung in der Quantengravitation steht auch in Beziehung zu [9] und [10].
Der zweite Teil behandelt homogene (pseudo-Riemannsche) Geometrien auf Mannigfaltigkeiten und deren Klassifikation nach Isometrien mit kosmologischen Anwendungen, auf Grundlage von [11], [12], und [13]. Die mögliche Anwendung in der multidimensionalen Kosmologie wurde auch schon in [14] dargestellt. Homogene Riemannsche Geometrien definiter Signatur und ihre Isometrien wurden bereits früher in [15], [16], [17], und [18] behandelt.
Der dritte Teil, über multidimensionale Geometrien, die Reduktion ihrer Einstein-Hilbert Wirkung zu einem effektiven σ-Modell, sowie über dessen Anwendung zur Gewinnung von Lösungen der multidimensionalen Gravitation und Kosmologie ist das Ergebnis von [19], [20], [21], [22], [23], [24], [25], [26], [27], und [28]. In Bezug hierzu stehen auch die frühen Arbeiten zur multidimensionalen Kosmologie [29], [30], [31], und zu konformen Transformationen [32], [33], [34].

2. *Zu kausalen Strukturen*
2.1. *Lokale Kegelstrukturen*
Schwache und starke lokale Kegelstrukturen wurden in beliebiger Dimension und für jede $\mathcal{C}^r$-Kategorie jeweils durch die Einbettung einer Kegelfläche bzw. einer mannigfaltigkeitartigen Verdickung einer solchen definiert.
2.2. *Kausalität auf topologischen Mannigfaltigkeiten*
Allgemeine Definitionen für kausale Strukturen wurden auf topologischen oder differenzierbaren $\mathcal{C}^r$-Mannigfaltigkeiten mittels geeigneter lokaler Kegelstrukturen gegeben.
2.3. *Kausale Indexmengen und Diffeomorphismen*
Die kompakten Doppelkegel definieren geeignete Indexmengen für die Konstruktion von kausalen Netzen in denen die Topologie der Mannigfaltigkeit kodiert ist. Dies ist für (global-hyperbolische) Lorentz-Mannigfaltigkeiten wohl bekannt. Topologisch-kausal definierte Doppelkegel ermöglichen die Konstruktion entsprechender Netze allgemeiner auf $\mathcal{C}^r$-Mannigfaltigkeiten. Mit einer durch lokale Kegel gegebene kausalen Struktur sind diejenigen Diffeomorphismen ausgezeichnet, die lokale Kegel wieder in solche abbilden.

3. *Zur kosmologischen Anwendung I: Quanten-Relativitätstheorie*

3.1. *Algebraische QFT auf Mannigfaltigkeiten*

Ein Teil der Haag-Kastler Axiome für algebraische Quanten(feld)theorie über dem Minkowski-Raum benutzt nur topologische Eigenschaften und Relationen der Indexmengen. Diese Axiome können direkt für differenzierbare Mannigfaltigkeiten verallgemeinert werden.

Der andere Teil der Haag-Kastler Axiome benutzt kausale Eigenschaften und Relationen der Indexmengen. Mittels des zuvor gewonnenen topologischen Kausalitätsbegriffes können auch diese Axiome für differenzierbare Mannigfaltigkeiten verallgemeinert werden.

3.2. *Quanten-Geometrie*

Die wichtigste aktuelle Anwendung findet das verallgemeinerte Haag-Kastler System der algebraische Quanten(feld)theorie in der Quantengeometrie der Einstein-Gravitation. Als besonders aktuelles Beispiel ist die Quantengeometrie der äusseren Umgebung eines kausalen Horizontes üeber jedem räumlichen Schnitt durch ein Netz von Weyl-Algebren für Zustände mit einer unendlichen Anzahl von Schnittpunkten von Kanten und transversalen Hyperflächen gegeben.

4. *Zu homogenen Strukturen*

4.1. *Homogene Mannigfaltigkeiten*

Homogenität einer Mannigfaltigkeit mit Struktur s liegt ganz allgemein dann vor, wenn die Struktur zwischen zwei Punkten durch einen s-erhaltenden Homeomorphismus transportiert wird. Die s-erhaltenden Homeomorphismen bilden bekanntlich eine Gruppe. Für (pseudo-)Riemannsche Mannigfaltigkeiten ist dies gerade die transitive Isometriegruppe.

4.2. *Lokale homogene Geometrien*

4.2.1. *Klassifizierende Räume lokaler Isometrien*

Jeder homogene (pseudo-)Riemannsche Raum trägt eine minimale transitive Isometriegruppe. Die Klassifikation der Lie-Algebren fester Dimension über algebraisch-geometrische Invarianten ist deshalb Bestandteil einer algebraisch-geometrischen Klassifikation homogener (pseudo-)Riemannsche Räume. Bis zur reellen Dimension $n = 4$ ist der klassifizierende (topologische) Raum K^n der zuhörigen Lie-Algebren bekannt.

4.2.2. *Zariski-duale Topologie und Lie-Algebra Kohomologie*

Die Topologie κ^n des Raumes K^n ist genau die duale Zariski-Topologie. Sie kann mittels der Lie-Algebra Kohomologie infinitesimaler Deformationen innerhalb der Kategorie der Lie-Algebren verstanden werden. In κ^n entsprechen abgeschlossene Punkte Lie-Algebren die keine spontanen Deformationen zulassen, und isolierte, abgeschlossene Punkte entsprechen Lie-Algebren die keine (weder spontane noch parametrische) Deformationen zulassen. Letztere haben verschwindende, zweite Lie-Algebra Kohomologie H^2.

4.2.3. *Klassifizierende Räume lokaler homogener Geometrien*

Für lokale homogene 3-dimensionale Riemannsche Geometrien definiter Signatur existiert eine kanonische Konstruktion eines klassifizierenden Raumes mittels algebraischer Invarianten des Ricci-Tensors. Für den analogen Fall Lorentzscher Signatur konnten bisher nur partielle Resultate gewonnen werden. Die indefinite Signatur führt hier zu Entartungen in den algebraischen Invarianten. Im Prinzip müssten im Tangentialraum sämtliche relativen Orientierungen der Lichtkegelachse gegenüber dem Ellipsoid der anisotropien Skalen betrachtet werden. Die vorliegende Untersuchung von drei Permutationen der Lorentz-Signatur ergibt für den von den entsprechenden algebraischen Invarianten aufgespannten Raum jeweils qualitativ verschiedene Resultate.

5. *Zur kosmologischen Anwendung II: Rigität von Isometrien*
Im Falle 3-dimensionaler lokaler homogener Riemannscher Geometrien zerfällt deren klassifizierender Raum in Untermannigfaltigkeiten fester charakteristischer 3-dimensionaler Isometrie. Eine homologische Klassifikation dieser Untermannigfaltigkeiten ergibt, daß spontane Transitionen (bzw. Deformationen) zwischen den Lie-Algebren der Isometrien genau solche sind, die die Dimension der zugehörigen Untermannigfaltigkeit im klassifizierenden Raum erniedrigen (bzw. erhöhen). Die Erniedrigung der Dimension findet stets auf dem Rand der entsprechenden Untermannigfaltigkeit statt. Die lokale Isometrie eines inneren Punktes einer Untermannigfaltigkeit maximaler Dimension ist dementsprechend stabil unter Deformationen der zugehörigen Geometrie.

6. *Zu multidimensionalen Strukturen*
6.1. *Das effektive σ-Modell der reinen multidimensionalen Geometrie*
Die Einstein-Hilbert Wirkung für eine multidimensionale pseudo-Riemannsche Geometrie mit homogenen Faktorräumen und geeigneten Randbedingungen reduziert sich im Falle räumlich homogener oder statischer Geometrien auf ein mechanisches System über einem Minkowskischen Minisuperraum. Wie in [22] erstmals gezeigt, ist eine solche Wirkung im allgemeineren Fall homogener interner Faktorräume, bei geeigneten Randbedingungen, stets auf ein effektives σ-Modell von einer (nicht notwendig homogenen) Mannigfaltigkeit (M_0, g_0) in einen konform-flachen, homogenen Target-Raum zurückführbar.
6.2 *Das σ-Model mit extra Skalaren und $p+2$-Formen*
Wie erstmals in [27] gezeigt wurde, kann auch eine um zusätzliche Skalarfelder und antisymmetrische $p+2$-Formen (in physikalisch typischer Weise) erweiterte Wirkung auf ein effektives σ-Modell von einer inhomogenen Mannigfaltigkeit (M_0, g_0) in einen erweiterten Target-Raum reduziert werden.
6.3 *Struktur des Targetraumes*
Wie in [19] gezeigt, ist der Target-Raum stets ein homogener Raum. Er ist darüber hinaus genau dann lokal symmetrisch, wenn die Kopplungsvektoren der Felder entweder orthogonal oder entartet sind.
6.4 *Spezielle Koordinateneichungen auf M_0*
Spezielle Koordinaten-Eichungen einer flache Basismannigfaltigkeit (M_0, g_0) der multidimensionalen Geometrie (M, g) sind in der Literatur weit verbreitet. Die am häufigsten verwendeten Koordinaten sind die sogenannten *Eigenkoordinaten*, in denen die Metrik g_0 eine Standarddiagonalform annimmt, und die *harmonischen Koordinaten*, auf denen der Laplace-Beltrami-Operator der multidimensionalen Metrik g verschwindet. Für $\dim M_0 = 1$ heißt die entsprechende Koordinate *Eigenzeit* bzw. *harmonische Zeit*.

7. *Zur kosmologischen Anwendung III: Multidimensionale Lösungen*

Die Literatur über multidimensionale Gravitation und Kosmologie umfaßt zahlreiche Untersuchungen zu den stationären Punkten einer Einstein-Hilbert-Wirkung für multidimensionale Geometrien mit optional weiteren angekoppelten Feldern. Die meisten dieser auch kurz *Lösungen* genannten stationären Punkte ergeben sich als Spezialfälle allgemeiner Lösungen zum effektiven σ-Modells der multidimensionalen Geometrie, gegebenfalls mit einem um entsprechende, optionale Zusatzfelder erweiterten Target-Raum. 7.1. *Lösungen mit Abelschem Targetraum*

Für dim $M_0 \neq 2$ wurde eine neue Ricci-flache Lösung für das multidimensionale σ-Modell mit Skalarfeldern gefunden, die verallgemeinertes inflationäres Verhalten, nicht nur in zeitartiger Richtung, sondern auch in zusätzlichen raumartigen Richtungen von M_0 zuläßt.

7.2. *Orthobran-Lösungen mit* ${}^{(E)}V = 0$

Orthobrane sind p-Brane deren Kopplungsvektoren eine bestimmte Orthogonalitätsbedingung erfüllen. Diese stellt für feste Kopplungen eine topologische Bedingung an die Dimensionen der Schnittmannigfaltigkeiten verschiedener Brane dar. Lösungen des multidimensionalen σ-Models mit Orthobranen und verschwindendem, multidimensionalen Potential ${}^{(E)}V$ können allgemein aus einer entsprechenden Anzahl von harmonischen Funktionen über (M_0, g_0) gewonnen werden.

7.3. *Spherisch-symmetrische p-Brane*

Im Falle des σ-Modells statischer, spherisch-symmetrischer, multidimensionaler Geometrien mit Ricci-flachen, internen Faktorräumen und p-Branen, und Skalarfeld reduzieren sich die Feldgleichungen auf die eines Euklidischen Toda-Systems.

7.4. *Schwarze Löcher mit EM Branen*

Für sich schneidende, elektische (E) und magnetische (M) Brane in einer minimalen Konfiguration mit beliebiger Dimensionen wurde in zwei Fällen jeweils eine Schar von Lösungen des statischen, spherisch-symmetrischen multidimensionalen σ-Modells gefunden:

1. falls die EM Brane Orthobrane sind,
2. falls die Ladungen der p-Brane degeneriert sind, $Q_e^2 = Q_m^2$.

Die qualitativen Eigenschaften dieser Lösungen sind in beiden Fällen nicht von den Dimensionen der einzelnen Faktorräume abhängig, sondern lediglich von der Dimension $d_2 + 1$ des Schnitts der beiden p-Brane. In beiden Fällen sind die Lösungen parametrisiert durch die beiden Ladungen Q_e und Q_m der p-Brane sowie einen weiteren (masseartigen) Parameter k, von dem insbesondere die Hawking-Temperatur T_H für den (scheinbaren) Horizont eines schwarzen Loches abhängt. Im Limes $k \to 0$ skaliert T_H in beiden Fällen (mit nicht verschwindenden Ladungen) für $d_2 = 0$ gegen Null, für $d_2 = 1$ gegen einen endlichen Wert, und sonst gegen Unendlich.

Die bekannten Reissner-Nordström Lösungen ergeben sich aus der 2. Schar von Lösungen, für $d_2 = 0$.

7.5. *Räumlich homogene Lösungen*

Multidimensionale kosmologische Modelle mit Ricci-flachen Faktorräumen und komponentigem perfektem Fluid ergeben sich als Spezialfälle aus dem σ-Modell der multidimensionalen Geometrie, wobei jede Komponente durch genau ein zusätzliches Skalarfeld beschrieben wird. Die allgemeine Dynamik eines solchen, anisotropen Modells ist quali-

tativ durch ein Kasner-artiges Verhalten nahe der Singularität und eine Isotropisierung während der Expansion gegen in eine asymptotische de Sitter-Raumzeit charakterisiert.
Für ein integrables 3-Komponenten-Modell (mit steifer Materie, Staub, und Vakuum) wurden die expliziten klassische Lösungen mit Lorentzscher Signatur, sowie die analogen Lösungen Euklidischer Signatur bestimmt. Aus letzteren wurden klassische Wurmlöcher konstruiert. Die Potentiale der Skalarfelder können nach einem allgemeinen Verfahren für das integrable Modell explizit rekonstruiert werden. Für die Wheeler-de Witt-Gleichung des quantisierten Modells gibt es ebenfalls exakte Lösungen, sowohl mit diskretem als auch kontinuierlichem Spektrum.

7.6. *Das Einsteinsche Bezugssystem in der Kosmologie*

Physikalisch relevante kosmologische Lösungen sollten stets im Einsteinschen Bezugsystem, definiert durch minimale Kopplung der zusätzlichen Felder an die Metrik, interpretiert werden. Die konforme Transformation von Lösungen des multidimensionalen σ-Models in entsprechende Lösungen im Einsteinsche Bezugsystem kann für $\dim M_0 \neq 2$ allgemein angegeben werden. In konkreten Beispielen hat die transformierte Lösung im allgemeinen qualitativ wesentlich verschiedene Eigenschaften. Es wurde insbesondere gezeigt, daß urprünglich inflationäre Lösungen dann im Einsteinschen Bezugssystem oft nicht mehr inflationär sind.

7.7. *Multidimensionale m-komponentige Kosmologie*

Das multidimensionale kosmologische Modell mit m-komponentigem perfektem Fluid mit verschiedenen Drucken in n Ricci-flachen Faktorräumen ($2 \leq m \leq n-1$) kann durch bestimmte Materie-Vektoren charakterisiert werden. Im Falle, daß diese die Wurzelvektoren einer Lie-Algebra vom Cartan-Typ A_m sind, reduziert sich das Modell auf eine klassische offene Toda-Kette. Diese wurde nach einem in der Literatur bekannten Verfahren integriert. Für ein Beispiel mit $n = 3$ und $m = 2$ wurde die Lösung qualitativ diskutiert und ins Einsteinsche Bezugssystem transformiert.

7.8. 2-*dimensionale Dilaton-Gravitation*

Im Falle $\dim M_0 = 2$ entspricht das multidimensionale σ-Modell einer reinen Dilaton-Gravitation. Für die Reduktion einer $2+3$-dimensionalen Geometrie mit spherischer Symmetrie wurden statische Lösungen bestimmt und das Horizont-Problem diskutiert.

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