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to the memory of
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Calibrated Stochastic Pricing Models: From Interest to Credit Derivatives

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VALUE & RISK

ENAMEC

1. **Interest Product Valuation**

- Stochastic Interest Models
- Short Rate Models
- Pricing of IR Derivatives
- Market Forward Rate Models

2. **Credit Derivatives Valuation**

- Stochastic Credit Default Models
- Pricing Credit Default Swaps
- Stochastic Intensity Models
- CDS Options
- CDS Rate Models

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given: $df_{T|t}$ term structure of zero bonds (discount factors) at time *t*

→ notions of interest rates:

- instantaneous forward rate

$$f(t,T) \quad := \quad -\frac{\partial}{\partial T} \ln df_{T|t}$$

- short rate

$$\begin{aligned} r(t) \quad &:= \quad - \lim_{T \to t_+} \frac{\partial}{\partial T} \ln df_{T|t} \\ &= \quad \lim_{T \to t_+} f(t,T) \quad , \end{aligned}$$

- (continuously compounded) zero rate

$$R(t,T) \quad = \quad \frac{1}{T-t} \int_t^T f(t,s) ds$$

- forward rate

$$\begin{aligned} R(T,T'|t) \quad &:= \quad \frac{1}{T'-T} \ln \frac{df_{T'|t}}{df_{T|t}} = \frac{R(t,T')(T'-t) - R(t,T)(T-t)}{T'-T} \\ &= \quad \frac{1}{T'-T} \int_T^{T'} f(t,s) ds \quad . \end{aligned}$$

Stochastic Interest Models

HJM framework → related descriptions of stochastic interest in terms of

- short rate $r(t)$
- instantaneous forward rates $f(t,T)$
- zero rates $R(t,T)$
- forward rates $R(T,T'|t)$

→ market rates (money market)

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Short Rate Models

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stochastic process for the short rate:

$$dr(t) = \mu(r(t),t)dt + \sigma(r(t),t)dW$$

- μ is the drift
- σ is the (local, deterministic) volatility
- W is a Brownian motion

→ zero bond prices:

$$P(t,T) = E_t[e^{-\int_t^T r(s)ds}]$$

Short Rate Models

stochastic process for the short rate:

$$dr(t) = \mu(r(t),t)dt + \sigma(r(t),t)dW$$

- μ is the drift ← calibration to term structure
- σ is the (local, deterministic) volatility
- W is a Brownian motion

→ zero bond prices:

$$P(t,T) = E_t[e^{-\int_t^T r(s)ds}]$$

Short Rate Models

comparison of popular short rate models:

model	dynamics	$\varphi \neq 0$	$r > 0$	$r \sim$	EBP	EOP
1-factor	$r_t = x_t + \varphi_t$					
V	$dr_t = a(\theta - r_t)dt + \sigma dW$	N	N	$\mathcal{N}$	Y	Y
HW	$dr_t = a(\theta_t - r_t)dt + \sigma dW$	Y	N	$\mathcal{N}$	Y	Y
BK	$d\ln r_t = (\eta_t - a\ln r_t)dt + \sigma dW$	Y	Y	$L\mathcal{N}$	N	N
CIR	$dr_t = a(\theta - r_t)dt + \sigma\sqrt{r_t}dW$	N	Y	$NC\chi^2$	Y	Y
CIR++	$dx_t = a(\theta - x_t)dt + \sigma\sqrt{x_t}dW$	Y	Y[φ]	$SNC\chi^2$	Y	Y
2-factor	$r_t = x_{1,t} + x_{2,t} + \varphi_t$					
G2/G2++	$dx_{i,t} = a_i(\theta_i - x_{i,t})dt + \sigma_i dW_i$ $i = 1, 2;\ dW_1 dW_2 = \rho dt$	N/Y	N	$\mathcal{N}$	Y	Y
LS/CIR2++	$dx_{i,t} = a_i(\theta_i - x_{i,t})dt + \sigma_i\sqrt{x_{i,t}}dW_i$ $i = 1, 2;\ dW_1 dW_2 = \rho dt$	N/Y	Y[φ]	$\Sigma NC\chi^2 /$ $\Sigma SNC\chi^2$	Y for $\rho = 0$	Y for $\rho = 0$

Short Rate Models

affine term structure models:

$$R(t,T) = \alpha(t,T) + \beta(t,T)r(t)$$

$$\begin{aligned} P(t,T) &= e^{-(\alpha(t,T)+\beta(t,T)r(t))(T-t)} \\ &=: A(t,T)e^{-B(t,T)r(t)} \quad . \end{aligned}$$

Lemma: If the short rate process

$$dr = \mu(t,r)dt + \sigma(t,r)dW$$

has affine coefficients,

$$\begin{aligned} \mu(t,r) &= \lambda(t)r + \eta(t) \\ \sigma(t,r)^2 &= \gamma(t)r + \delta(t) \end{aligned}$$

then, the model has an affine term structure.

examples of affine term structure models:

model	λ	η	γ	δ
V	$-a$	$a\theta$	0	σ^2
CIR	$-a$	$a\theta$	σ^2	0
HW	$-a$	$a\theta(t)$	0	σ^2
CIR++	$-a$	$a\theta(t)$	σ^2	0

functions A and B are related to the affine coefficients by

$$\frac{\partial B(t,T)}{\partial t} + \lambda(t)B(t,T) - \frac{1}{2}\gamma(t)B(t,T)^2 + 1 = 0, \quad B(T,T) = 0$$

$$\frac{\partial A(t,T)}{\partial t} - \eta(t)B(t,T) + \frac{1}{2}\delta(t)B(t,T)^2 = 0, \quad A(T,T) = 1$$

→ explicit solutions for particular coefficients, e.g. like V, CIR

Short Rate Models

Gaussian model (Vasicek):

$$dr = a(\theta - r(t))dt + \sigma dW_t$$

$$r(t) = r(s)e^{-a(t-s)} + \theta(1 - e^{-a(t-s)}) + \sigma \int_s^t e^{-a(t-u)} dW_u$$

moments given by:

$$E[r(t)|\mathcal{F}_s] = r(s)e^{-a(t-s)} + \theta(1 - e^{-a(t-s)})$$

$$V[r(t)|\mathcal{F}_s] = \frac{\sigma^2}{2a}[1 - e^{-2a(t-s)}]$$

term structure given by:

$$A(t,T) = e^{(\theta - \frac{\sigma^2}{2a})[B(t,T) - T + t] - \frac{\sigma^2}{2a} B(t,T)^2}$$

$$B(t,T) = \frac{1}{a}[1 - e^{-a(T-t)}]$$

Short Rate Models

explicit European option price:

$$
\begin{aligned}
PV_{call/put;ZB}(t,T,S,X) &= P(t,T)E^T[[\pm(P(T,S)-X)]^+|\mathcal{F}_T] \\
&= \pm P(t,T)[P(T,S)N(\pm d_+) - XN(\pm d_-)]
\end{aligned}
$$

$$
\begin{aligned}
d_\pm &:= \frac{1}{\sigma_p}\ln\frac{P(T,S)}{X} \pm \frac{\sigma_p}{2} \\
\sigma_p &:= \sigma\sqrt{\frac{1-e^{-2a(T-t)}}{2a}}B(T,S)
\end{aligned}
$$

essential ingredient for calibration to

- swaption quotes (volas)
- caplet quotes (volas)

Short Rate Models

CIR model:

$$dr = a(\theta - r(t))dt + \sigma\sqrt{r(t)}dW_t$$

1st and 2nd momentum given by:

$$\begin{aligned} E[r(t)|\mathcal{F}_s] &= r(s)e^{-a(t-s)} + \theta(1 - e^{-a(t-s)}) \\ V[r(t)|\mathcal{F}_s] &= r(s)\frac{\sigma^2}{a}[e^{-a(t-s)} - e^{-2a(t-s)}] + \theta\frac{\sigma^2}{2a}[1 - e^{-2a(t-s)}]^2 \end{aligned}$$

term structure given by:

$$\begin{aligned} A(t,T) = A^{CIR}(t,T) &= \left[\frac{2he^{(T-t)(h+a)/2}}{C(t,T)}\right]^{\frac{2a\theta}{\sigma^2}} \\ B(t,T) = B^{CIR}(t,T) &= \frac{2e^{(T-t)h} - 1}{C(t,T)} \\ C(t,T) &:= 2h + (h+a)e^{(T-t)h} - 1 \\ h &:= \sqrt{a^2 + 2\sigma^2} \end{aligned}$$

CIR - explicit European option price:

$$
\begin{aligned}
PV_{call/put} &= P(t,T)E^T[[\pm(P(T,S)-X)]^+|\mathcal{F}_t] \\
&= P(t,T)[P(T,S)(\chi^2(d_+;\nu,\lambda_+)-\frac{1}{2}\pm\frac{1}{2})-X(\chi^2(d_-;\nu,\lambda_-)-\frac{1}{2}\pm\frac{1}{2})]
\end{aligned}
$$

$$
d_\pm := \frac{2[\rho+\psi+B(T,S)(\frac{1}{2}\pm\frac{1}{2})]}{B(T,S)} \ln\frac{A(T,S)}{X}
$$

$$
\begin{aligned}
\nu &:= \frac{4a\theta}{\sigma^2} \\
\lambda_\pm &:= \frac{2\rho^2 r(t)e^{h(T-t)}}{\rho+\psi+B(T,S)(\frac{1}{2}\pm\frac{1}{2})}
\end{aligned}
$$

$$
\begin{aligned}
\rho &:= \frac{2h}{\sigma^2(e^{h(T-t)}-1)} \\
\psi &:= \frac{a+h}{\sigma^2}
\end{aligned}
$$

Short Rate Models

deterministic shift extension: $\rightarrow$ enables calibration to term structure

$$\begin{aligned} dx_t &= \mu(x_t;\theta)dt + \sigma(x_t;\theta)dW \\ r_t &= x_t + \varphi(t;\theta) \end{aligned}$$

θ are a collection of model parameters.

$$\rightarrow \quad dr_t = [\mu(r_t - \varphi(t;\theta);\theta) + \frac{d\varphi}{dt}]dt + \sigma(r_t - \varphi(t;\theta);\theta)dW$$

$$P^x(t,T;\theta) = E_x[e^{-\int_t^T x_s ds}|\mathcal{F}^x] \stackrel{EBP}{=} \Pi^x(t,T,x_t;\theta)$$

$$\begin{aligned} \rightarrow \quad P(t,T;\theta,\varphi) &= E_x[e^{-\int_t^T r_s ds}|\mathcal{F}^r] \\ &= e^{-\int_t^T \varphi(s;\theta)ds}\Pi^x(t,T,x_t;\theta) \end{aligned}$$

calibration: calibrating $P(0,T;\theta,\varphi) = P^M(0,T)$ is equivalent to calibrating

$$f^x(0,t;\theta) + \varphi(t;\theta) = f^M(0,t) \quad \longrightarrow \quad \varphi(t;\theta)$$

$$\begin{aligned} PV^x_{call/put}(t,T,S,X;\theta) &= E_x[e^{-\int_t^T x_s ds}[\pm(P^x(T,S) - K)]^+|\mathcal{F}^x] \\ &\stackrel{EOP}{=} \Psi^x_{c/p}(t,T,S,X,x_t;\theta) \quad . \end{aligned}$$

$$\rightarrow \quad PV_{call/put}(t,T,S,X;\theta,\varphi) = e^{-\int_t^S \varphi(s,\theta)ds}\Psi^x(t,T,S,Xe^{\int_T^S \varphi(s,\theta)ds}, r_t - \varphi(s,\theta);\theta)$$

option prices calibrated to term structure curve

Short Rate Models

shifted Gaussian model (Hull White):

$$dr = [\vartheta(t) - ar(t)]dt + \sigma dW_t$$

calibration to term structure:

$$\begin{aligned}\vartheta(t) &= (\frac{d}{dt} + a)\varphi(t) \\ &= (\frac{d}{dt} + a)(f^M(0,t) - f^x(0,t;a,\sigma)) \\ &= \frac{\partial f^M(0,t)}{\partial T} + af^M(0,t) + \frac{\sigma^2}{2a}[1 - e^{-2at}]\end{aligned}$$

calibration of (σ,a)

→ optimization problem:

minimize deviation of option prices $PV_{call/put}(t,T,S,X;\theta,\varphi)$

from quoted market values

(swaptions, caplets)

Short Rate Models

shifted CIR model (CIR++):

$$\begin{aligned} dx &= a(\theta - x(t))dt + \sigma\sqrt{x(t)}dW_t \quad ; \quad x(0) = x_0 \\ r(t) &= x(t) + \varphi(t) \end{aligned}$$

calibration to term structure:

$$\varphi(t;\theta) = f^M(0,t) - f^x(0,t;\theta)$$

$$f^x(0,t;\theta) = \frac{2a\theta(e^{ht}-1)}{2h+(h+a)e^{(T-t)h}-1} + x_0\frac{4h^2e^{ht}}{2h+(h+a)e^{(T-t)h}-1}$$

calibration of (σ,a)

→ optimization problem:

minimize deviation of option prices from quoted market values (swaptions, caplets)

$$PV_{call/put}(t,T,S,X;\theta,\varphi)$$

Short Rate Models

calibration of vola and mean reversion parameters

$$\theta := (a, \sigma) \in \mathbb{R}^+ \times \mathbb{R}^+ =: S$$

$$\begin{aligned} d^2(\theta) &:= ||\mathbf{Q}_{\mathrm{model}}(\theta) - \mathbf{Q}_{\mathrm{target}}||_2^2 \\ &= \sum_{i=1}^{n} (PV_{\mathrm{HW},i}(a, \sigma) - Q_{\mathrm{Black},i})^2 \end{aligned}$$

inverse problem: find (σ,a) such that $\min_{\theta \in S} d(\theta)$ is obtained

→ use e.g. an adaptive lattice algorithm with logarithmic scales

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Forward Rate Agreements (FRAs)

$$
\begin{aligned}
\mathrm{Payoff}_{FRA}(S,T) &= N\tau(S,T)(X - L(S,T)) \\
&= N\left[\underbrace{\tau(S,T)X + 1}_{=:B} - \frac{1}{P(S,T)}\right]
\end{aligned}
$$

$$
\begin{aligned}
PV_{FRA}(t,S,T) &= NP(t,T)\tau(S,T)(X - \underbrace{E_t[L(S,T)]}_{=:F(t,S,T)}) \\
&= N\left[B \cdot P(t,T) - P(t,S)\right] \quad ,
\end{aligned}
$$

forward rate:

$$
F(t,S,T) = \frac{1}{\tau(S,T)}\left(\frac{P(t,S)}{P(t,T)} - 1\right)
$$

the rate X which makes the FRA fair at time 0 numeraire $P(t,T)$

→ forward martingale measure Q^T

$$\frac{X_t}{P(t,T)} = E^T\left[\frac{X_u}{P(u,T)}|\mathcal{F}_t\right] \quad ,$$

for any tradable asset X, in particular also for

$$X(t) := \frac{1}{\tau(S,T)}(P(t,S) - P(t,T)) \quad .$$

Swaps

$$\text{Payoff}_{RIRS/PIRS} = \pm \sum_{i=1}^{n} N\tau_i (X - L(T_{i-1}, T_i))$$
$$= \pm \sum_{i=1}^{n} \text{Payoff}_{FRA}(T_{i-1}, T_i)$$

$$PV_{RIRS/PIRS}(t; \mathbf{T}, \tau) = \pm \sum_{i=1}^{n} PV_{FRA}(t, T_{i-1}, T_i)$$
$$= \pm N \sum_{i=1}^{n} P(t, T_i)\tau_i (X - F(t, T_{i-1}, T_i))$$
$$= \pm \left[\underbrace{NP(t, T_n) + NX \sum_{i=1}^{n} P(t, T_i)\tau_i}_{PV_{Bond}(t;\mathbf{T},\tau)} - \underbrace{NP(t, T_0)}_{PV_{FRN}(t;\mathbf{T},\tau)} \right]$$
$$= \pm N \sum_{i=1}^{n} P(t, T_i)\tau_i \left[X - \underbrace{\frac{P(t, T_0) - P(t, T_n)}{\sum_{i=1}^{n} P(t, T_i)\tau_i}}_{=:S(t;T_0,T_n)} \right],$$

forward swap rate

$$S_{0,n}(t) = S(t;T_0,T_n) = \frac{P(t,T_0) - P(t,T_n)}{\sum_{i=1}^{n} P(t,T_i)\tau_i}$$

$$= \left(1 - \frac{P(t,T_n)}{P(t,T_0)}\right)\left(\sum_{i=1}^{n}\tau_i \frac{P(t,T_i)}{P(t,T_0)}\right)^{-1}$$

$$= \frac{1 - \prod_{j=1}^{n}\frac{1}{1+\tau_j F_j(t)}}{\sum_{i=1}^{n}\tau_i \prod_{j=1}^{i}\frac{1}{1+\tau_j F_j(t)}} \quad , \; F_j(t) := F(t;T_{j-1},T_j)$$

the rate X which makes the swap fair at time 0

numeraire $\quad C(t) := \sum_{i=1}^{n} P(t,T_i)\tau_i$

→ swap martingale measure

$$\frac{X_t}{C_t} = E^C\left[\frac{X_T}{C_T}|\mathcal{F}_t\right] \quad ,$$

for any tradable asset X, in particular also for

$$X(t) := P(t,T_0) - P(t,T_n) \quad .$$

swap rate in terms of forward rates

$$S_{0,n}(t) = \frac{\sum_{i=1}^{n} \tau_i P(t,T_i) F_i(t)}{\sum_{i=1}^{n} \tau_i P(t,T_i)}$$

$$= \sum_{i=1}^{n} w_i(t) F_i(t)$$

$$w_i(t) := \frac{\tau_i P(t,T_i)}{\sum_{i=1}^{n} \tau_i P(t,T_i)}$$

$$= \tau_i \frac{\prod_{j=1}^{i} \frac{1}{1+\tau_j F_j(t)}}{\sum_{i=1}^{n} \tau_i \prod_{j=1}^{i} \frac{1}{1+\tau_j F_j(t)}}$$

linear approximation:

$$w_i(t) \approx w_i(0)$$

Pricing of IR Derivatives

European Bond Options

$$\text{Payoff}_{call/put;B}(t,T;\mathbf{T},\tau) = \left[\pm(\sum_{i=1}^{n} P(T,T_i)C_i - X)\right]^{+}$$
$$= \left[\pm(\sum_{i=1}^{n} C_i \Pi(T,T_i,r(T)) - X)\right]^{+}$$

Jamshidian Lemma (1989):
Let r^* be a solution of

$$\sum_{i=1}^{n} C_i \underbrace{\Pi(T,T_i,r^*)}_{=:X_i} = X \quad .$$

Assume that the analytic zero bond price satisfies the monotonicity

$$\frac{\partial \Pi(T_1,T_2,r)}{\partial r} < 0, \quad \text{for } 0 < T_1 < T_2 \quad .$$

Then $\left[\pm(\sum_{i=1}^{n} C_i \Pi(T,T_i,r(T)) - X)\right]^{+} = \sum_{i=1}^{n} C_i \left[\pm(\Pi(T,T_i,r(T)) - \Pi(T,T_i,r^*))\right]^{+}$

Monotonicity condition is satisfied for affine term structure with A,B>0
→ HW, CIR++ applicable

$$PV_{call/put;B}(t,T;\mathbf{T},\tau) = \sum_{i=1}^{n} C_i PV_{call/put;ZB}(t,T,T_i,X_i)$$

Swaptions

$$\text{Payoff}_{RS/PS} = N\left[\mp\sum_{i=1}^{n} P(T_0,T_i)\tau_i(F(T_0,T_{i-1},T_i)-X)\right]^+$$

$$= N\left[\pm(\sum_{i=1}^{n} P(T_0,T_i)C_i - 1)\right]^+$$

$$C_i = X\tau_i \quad \text{for } i = 1,\ldots,n-1$$

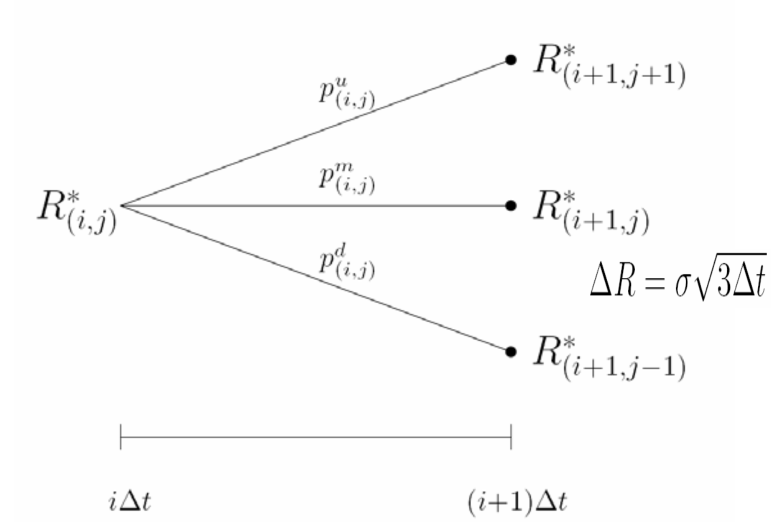
$$C_n = X\tau_n + 1$$

$$PV_{RS/PS;B}(t,T;\mathbf{T},\tau) = \sum_{i=1}^{n} C_i PV_{call/put;ZB}(t,T,T_i,X_i)$$

Pricing of IR Derivatives

Bermudean bond options / swaptions: → **tree algorithm** e.g. trinomial

1. use tree approximation for a time-homogeneous short rate process (V,CIR):



$$dr^*(t) = -ar^*(t)\,dt + \sigma\,dW(t)$$

2. determine branching probabilities from the moments

$$\begin{aligned}
E[r(t)|\mathcal{F}_s] \;\rightarrow\; p^u_{(i,j)}\Delta R + p^d_{(i,j)}(-\Delta R) &= -aj\Delta R\Delta t \\
V[r(t)|\mathcal{F}_s] \;\rightarrow\; p^u_{(i,j)}\Delta R^2 + p^d_{(i,j)}\Delta R^2 &= \sigma^2\Delta t + a^2 j^2 \Delta R^2 \Delta t^2 \\
p^u_{(i,j)} + p^m_{(i,j)} + p^d_{(i,j)} &= 1
\end{aligned}$$

Pricing of IR Derivatives

tree algorithm e.g. trinomial

2. determine tree-approximation for the calibrated shift extended short rate process (HW,CIR++):

$\rightarrow$ calibrate the shift $\alpha_i = R_{(i,j)} - R^*_{(i,j)}$

to the market term structure P(0,T):

$$P(0,(i+2)\Delta t) \stackrel{!}{=} e^{-\alpha_{i+1}\Delta t} \sum_{k=-(i+1)}^{i+1} Q(i+1,k) e^{-k\Delta R \Delta t}$$

with

$$\begin{aligned} Q(i+1,j) &= p^u_{(i,j-u)} Q(i,j-u) e^{-(\alpha_i + (j+u)\Delta R)\Delta t} \\ &\quad + p^m_{(i,j-m)} Q(i,j-m) e^{-(\alpha_i + (j+m)\Delta R)\Delta t} \\ &\quad + p^d_{(i,j-d)} Q(i,j-d) e^{-(\alpha_i + (j+d)\Delta R)\Delta t} \end{aligned}$$

3. determine the underlying prices $P_{(i,j)}(s)$ on the tree

4. evaluate the derivative backward through the tree

summary:

bonds, FRAs, swaps:
can be priced as linear combination of zerobonds
with respect to an appropriate numeraire

European bond options, swaptions:
can be priced as linear combination of options on zerobonds

caps/floors:
can be priced as linear combination of European options on forward rates

Bermudean options/swaptions:
can be priced using a tree-approximation for the short rate process

exotic derivatives with strongly path-dependent payoff:
need simulation → simulation of market (LIBOR) forward rates

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Market Forward Rate Models

most important versions:

• LIBOR market model (Brace Gatarek Musiela 1997)
based on a multi-dim. stochastic process for the forward rates $F_j(t)$

• swap market model (Rebonato et al 1999)
based on a multi-dim. stochastic process for the swap rates $S_{0,n}(t)$

problem: log-normal assumption can not be satisfied for both
forward rates and swap rates
but:
relevant quoted volas: caplets and swaptions

nevertheless: consider **log-normal forward rate model**

$$dF_i = \sigma_i(t) F_i(t) dZ_i$$

$$dZ_i dZ_j = \rho_{ij} dt$$

(using the forward martingale measure)

log-normal forward rate model

Black swaption variance is defined as

$$(v_{0,n}(T_0))^2 \;\; := \;\; \int_0^{T_0} \sigma_{0,n}(t)dt = \int_0^{T_0} (d\ln S_{0,n}(t))(d\ln S_{0,n}(t))$$

$$\begin{aligned}
dS_{0,n}(t) &= \sum_{i=1}^{n} (w_i(t)dF_i(t) + F_i(t)dw_i(t) + (\ldots)dt \\
&= \sum_{i,k=1}^{n} (w_k(t)\delta_{ik}dF_k(t) + F_i(t)dw_i(t) + (\ldots)dt \\
&= \sum_{k=1}^{n} (w_k(t) + \sum_{i=1}^{n} F_i(t)\frac{\partial w_i(t)}{\partial F_k})dF_k(t) + (\ldots)dt \\
&= \sum_{k=1}^{n} \bar{w}_k(t)dF_k(t) + (\ldots)dt \quad , \\
\bar{w}_k(t) &:= w_k(t) + \sum_{i=1}^{n} F_i(t)\frac{\partial w_i(t)}{\partial F_k} \quad .
\end{aligned}$$

Market Forward Rate Models

Hull White approximation of the swaption variance:

$$(v_{0,n})^2 \approx \sum_{i,j=1}^{n} \frac{\bar{w}_i(0)\bar{w}_j(0)F_i(0)F_j(0)\rho_{ij}}{S_{0,n}(0)^2} \int_0^{T_0} \sigma_i(t)\sigma_j(t)dt$$

Rebonato approximation of the swaption variance:

$$(v_{0,n})^2 \approx \sum_{i,j=1}^{n} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{ij}}{S_{0,n}(0)^2} \int_0^{T_0} \sigma_i(t)\sigma_j(t)dt$$

Rebonato approximation for the terminal correlation of forward rates

$$(Corr(F_i(T_0), F_j(T_0)) \approx \rho_{ij} \frac{\int_0^{T_0} \sigma_i(t)\sigma_j(t)dt}{\int_0^{T_0} \sigma_i(t)dt \int_0^{T_0} \sigma_j(t)dt}$$

→ when calibrating lognormal forward rate model swaptions consider terminal correlations of the forward rates

1. Interest Product Valuation

- Stochastic Interest Models
- Short Rate Models
- Pricing of IR Derivatives
- Market Forward Rate Models

2. Credit Derivatives Valuation

- **Stochastic Credit Default Models**
- Pricing Credit Default Swaps
- Stochastic Intensity Models
- CDS Options
- CDS Rate Models

Assumption 1:
The default process is assumed to be a (time-inhomogeneous) Poisson process N_t with *intensity* $\lambda(t)$ (*hazard rate*).

$$\Lambda(t) \quad := \quad \int_0^t \lambda(s)ds$$

$$N_t = P_{\Lambda(t)}$$

For a Poisson process N_t, the *default time* τ is defined as

$$\tau \quad := \quad inf\{t > 0 : N_t = 1\} \quad .$$

Lemma: Let N_t be a Poisson processes with cumulative hazard function Λ.

$$\xi \quad := \quad \Lambda(\tau)$$

is *independent* of λ. More specifically it is an exponential standard random variable, i.e. its cumulative distribution yields

$$Q[\xi \leq x] \quad = \quad 1 - e^{-x}$$

Assumption 2:
The intensity λ itself follows another stochastic process I_t which is independent of N_t.
In particular ξ and λ are independent random variables.

$$\rightarrow \quad Q[\tau \leq t] = E_\lambda [Q_\xi[\xi \leq \Lambda(t)]] = 1 - E_\lambda \left[e^{-\Lambda(t)}\right]$$

i.e. the *survival probability* is

$$Q[\tau > t] = E_\lambda \left[e^{-\int_0^t \lambda(s)ds}\right]$$

Under presence of default it is natural, to extend the default-free filtration $\mathcal{F}_t$ by monitoring of the default time as additional event, yielding the filtration

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \leq t)$$

Proposition: Payoff be the default-free payoff at maturity T

$$\rightarrow \quad E(1_{\tau>T}\text{Payoff}|\mathcal{G}_t) = \frac{1_{\tau>t}}{Q\{\tau > t|\mathcal{F}_t\}} E(1_{\tau>T}\text{Payoff}|\mathcal{F}_t)$$

Stochastic Credit Default Models

Definition:
A *defaultable zero bond (without recovery)* is defined by a payoff of $1_{\tau>T}$ at maturity T.

price of a defaultable zero bond (without recovery)

$$\begin{aligned}
\bar{P}(0,T) &:= E\left[D(0,T)1_{\tau>T}\right] \\
&= E\left[D(0,T)1_{\xi>\Lambda(T)}\right] \\
&= E\left[D(0,T)\underbrace{E[1_{\xi>\Lambda(T)}|\mathcal{F}_T]}_{=e^{-\Lambda(T)}}\right] \\
&= E\left[e^{-\int_0^T(r(T)+\lambda(T))}\right]
\end{aligned}$$

under the assumption that the processes for r and λ are uncorrelated

$$\bar{P}(0,T) = P(0,T)Q(\tau>t)$$

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Payoff variants of CDS:

1. Running CDS (RCDS):

The protection payment date is τ (payment at default).

The CDS contract holder (protection buyer) receives protection L_{gd} against default at τ if $T_0 < \tau \leq T_n$, and pays a charge with rate R at payment dates T_i, maximally until default time τ.

discounted payoff of the running protection leg (RPL) at time $t < T_0$

$$\begin{aligned} \text{Payoff}_{RPL;0,n}(t) &= 1_{T_0<\tau\leq T_n} D(t,\tau) L_{\text{gd}} \\ &= \sum_{i=1}^{n} 1_{T_{i-1}<\tau\leq T_i} D(t,\tau) L_{\text{gd}} \quad . \end{aligned}$$

Under default at time τ, the protection seller (receiver of the charge leg) obtains

$$\begin{aligned} \text{Payoff}_{RCDS;0,n}(t) &= D(t,\tau)(\tau - T_{\beta(\tau)-1}) R 1_{T_0<\tau<T_n} + \sum_{i=1}^{n} D(t,T_i)\tau_i R 1_{T_i\leq\tau} \\ &\quad -\text{Payoff}_{RPL;0,n}(t) \end{aligned}$$

Pricing Credit Default Swaps

artificial variant

Continuous Running Payment CDS (CRCDS):

$$\begin{aligned}\text{Payoff}_{CRCDS;0,n}(t) &= \int_{T_0}^{T} D(t,s) R 1_{s \leq \tau} ds \\ &\quad - \text{Payoff}_{RPL;T_0,T}(t)\end{aligned}$$

2. Postponed Payment Running CDS (PRCDS):

The protection payment date is $T_{\beta(\tau)}$ (postponed payment).
In case of default at τ with $T_0 < \tau \leq T_n$, the CDS contract holder (protection buyer) receives protection L_{gd} postponed at $T_{\beta(\tau)}$, and pays a charge with rate R at payment dates T_i, maximally until default time τ.

$$\text{Payoff}_{PRPL;0,n}(t) = \sum_{i=1}^{n} 1_{T_{i-1}<\tau\leq T_i} D(t, T_i) L_{\text{gd}}$$

At $T_{\beta(\tau)}$, the first payment date after the default, the CDS contract holder (protection buyer) receives protection L_{gd} against default if $T_0 < \tau \leq T_n$, and pays a charge with rate R at payment dates T_i with $i < \beta(\tau)$ (PRCDS-) or $i \leq \beta(\tau)$ (PRCDS+).

$$\text{Payoff}_{PRCDS-;0,n}(t) = \sum_{i=1}^{n} D(t, T_i)\tau_i R 1_{T_i\leq\tau} - \text{Payoff}_{PPL;0,n}(t)$$

$$\text{Payoff}_{PRCDS+;0,n}(t) = \sum_{i=1}^{b} D(t, T_i)\tau_i R 1_{T_{i-1}<\tau} - \text{Payoff}_{PPL;0,n}(t)$$

General pricing

$$
\begin{aligned}
PV_{CDS;0,n}(t;R,L_{\text{gd}}) &:= PV_{RCDS;0,n}(t;R,L_{\text{gd}}) = E[\text{Payoff}_{RCDS;0,n}(t)|\mathcal{G}_t] \\
&= \frac{1_{\tau>t}}{Q\{\tau>t|\mathcal{F}_t\}} E[\text{Payoff}_{RCDS;0,n}(t)|\mathcal{F}_t] \\
&= \frac{1_{\tau>t}}{Q\{\tau>t|\mathcal{F}_t\}} \Bigg\{ R \underbrace{E[D(t,\tau)(\tau - T_{\beta(\tau)-1}) 1_{T_0<\tau<T_n}|\mathcal{F}_t]}_{=:\text{accrual}_t} \\
&\qquad + \sum_{i=1}^{n} \tau_i R E[D(t,T_i) 1_{T_i \le \tau}|\mathcal{F}_t] - E[\text{Payoff}_{RPL;0,n}(t)|\mathcal{F}_t] \Bigg\} \\
&= \frac{1_{\tau>t}}{Q\{\tau>t|\mathcal{F}_t\}} \Bigg\{ R \left[\text{accrual}_t + \sum_{i=1}^{n} \tau_i E[D(t,T_i) 1_{T_i \le \tau}|\mathcal{F}_t] \right] \\
&\qquad - L_{\text{gd}} E[1_{T_0<\tau\le T_n} D(t,\tau)|\mathcal{F}_t] \Bigg\}
\end{aligned}
$$

Accordingly, the CDS forward rate $R_{0,n}(t)$ is the value R^* of the rate R such that the CDS is fair at contract time t, i.e. such that $PV_{CDS;0,n}(t; R_{0,n}(t), L_{\mathrm{gd}}) = 0$.

$$R_{0,n}(t) = \frac{L_{\mathrm{gd}} E[1_{T_0<\tau\leq T_n} D(t,\tau)|\mathcal{F}_t]}{\mathrm{accrual}_t + \sum_{i=1}^{n} \tau_i Q(T_i \leq \tau|\mathcal{F}_t)\bar{P}(t,T_i)}$$

Likewise, for a PRCDS the market rate is

$$R_{0,n}(t)^{PR} = \frac{L_{\mathrm{gd}} \sum_{i=1}^{n} E[D(t,T_i)1_{T_{i-1}<\tau\leq T_i}|\mathcal{F}_t]}{\sum_{i=1}^{n} \tau_i E[D(t,T_i)1_{\tau>T_{i-\iota}}|\mathcal{F}_t]} ,$$

where $\iota = 0$ for PRCDS- and $\iota = 1$ for PRCDS+.

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Pricing Credit Default Swaps

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Pricing under independence of interest rates and default times

$$
\begin{aligned}
\frac{PV_{RPL;0,n}(0)}{L_{gd}} &= PV_{RPL1;0,n}(0) := E[D(0,\tau)1_{T_0<\tau<T_n}] \\
&= E\left[\int_0^\infty D(0,t)1_{\tau\in[t,t+dt]}1_{T_0<\tau<T_n}\right] \\
&= \int_{T_0}^{T_n} E\left[D(0,t)\right] E\left[1_{\tau\in[t,t+dt]}\right] \\
&= \int_{T_0}^{T_n} P(0,t)\underbrace{Q(\tau\in[t,t+dt])}_{=-\underbrace{\frac{dQ(\tau\geq t)}{dt}}_{<0}dt} \\
&= -\int_{T_0}^{T_n} P(0,t)\frac{d}{dt}Q(\tau\geq t)dt
\end{aligned}
$$

Pricing Credit Default Swaps

Similarly

$$
\begin{aligned}
\text{accrual}_0 &:= E[D(0,\tau)(\tau - T_{\beta(\tau)-1})1_{T_0<\tau<T_n}] \\
&= \int_{T_0}^{T_n} E\left[D(0,t)\right](t - T_{\beta(t)-1})E\left[1_{\tau\in[t,t+dt]}\right] \\
&= \int_{T_0}^{T_n} P(0,t)(t - T_{\beta(t)-1})Q(\tau\in[t,t+dt]) \\
&= -\int_{T_0}^{T_n} P(0,t)(t - T_{\beta(t)-1})\frac{d}{dt}Q(\tau\geq t)dt
\end{aligned}
$$

and for the charge leg it holds

$$
\begin{aligned}
\frac{PV_{RCL;0,n}(0)}{R} &= PV_{RCL1;0,n}(0) \\
&:= \text{accrual}_0 + \sum_{i=1}^{n} E[D(0,T_i)]\tau_i E[1_{T_i\leq\tau}] \\
&= \text{accrual}_0 + \sum_{i=1}^{n} P(0,T_i)\tau_i Q(\tau\geq T_i) \quad .
\end{aligned}
$$

→ all determined by $P(0,\cdot)$ and $Q(\tau\geq\cdot)$

bootstrapping survival probabilities from CDS quotes

one solves

$$\begin{aligned} 0 &\stackrel{!}{=} PV_{CDS;0,n}(0; R^M_{0,n}(0), L_{\mathrm{gd}}) \\ &= R^M_{0,n}(0) PV_{RCL;0,n}(0) - L_{\mathrm{gd}} PV_{RPL;0,n}(0) \end{aligned}$$

successively with $T_n = n$ (in years) for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.

bootstrappng survival probabilities from CDS quotes

one solves

$$
\begin{aligned}
0 &\stackrel{!}{=} PV_{CDS;0,n}(0; R^M_{0,n}(0), L_{\mathrm{gd}}) \\
&= R^M_{0,n}(0) PV_{RCL;0,n}(0) - L_{\mathrm{gd}} PV_{RPL;0,n}(0)
\end{aligned}
$$

successively with $T_n = n$ (in years) for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.$

Implied hazard rates from CDS quotes:

$$Q[\tau > t] = e^{-\Gamma(t)}$$

$$Q(\tau \in [t, t+dt]) = \gamma(t) e^{-\Gamma(t)} dt$$

Usually for $n = 1, 2, 3, 5, 7, 10$ the CDS rates $R^M_{0,n}(0)$ are quoted. Between these maturities γ is usually assumed to interpolate in a specific form, e.g. linear or piecewise constant.

now consider piecewise constant γ.

$$\gamma(t) = \gamma_i \quad \text{for} \quad t \in [T_{i-1}, T_i[\quad ,$$

$$\Gamma(t) := \int_0^t \gamma(s) ds = \underbrace{\sum_{i=1}^{\beta(t)-1} (T_{i+1} - T_i)\gamma_i}_{=\Gamma_{\beta(t)-1}} + (t - T_{\beta(t)-1})\gamma_\beta(t)$$

$$\Gamma_j := \Gamma(T_j) = \sum_{i=1}^{j-1} (T_{i+1} - T_i)\gamma_i \quad .$$

Pricing Credit Default Swaps

$$
\begin{aligned}
PV_{CDS;0,n}(0;R,L_{\text{gd}};\Gamma(\cdot)) &= -R\int_{T_0}^{T_n} P(0,t)(t-T_{i-1})(-\gamma(t))e^{-\Gamma(t)}dt \\
&\quad +R\sum_{i=1}^{n} P(0,T_i)\tau_i e^{-\Gamma_i} \\
&\quad +L_{gd}\int_{T_0}^{T_n} P(0,t)(t-T_{i-1})(-\gamma(t))e^{-\Gamma(t)}dt \\
&= R\sum_{i=1}^{n}\gamma_i\int_{T_{i-1}}^{T_i} e^{-\Gamma_{i-1}-\gamma_i(t-T_{i-1})}P(0,t)(t-T_{i-1})dt \\
&\quad +R\sum_{i=1}^{n} P(0,T_i)\tau_i e^{-\Gamma_i} \\
&\quad -L_{gd}\sum_{i=1}^{n}\gamma_i\int_{T_{i-1}}^{T_i} e^{-\Gamma_{i-1}-\gamma_i(t-T_{i-1})}P(0,t)dt
\end{aligned}
$$

solve sucessively

$$0 \overset{!}{=} PV_{CDS;0,1}(0;R^M_{0,1},L_{\text{gd}};\gamma_1=\gamma_2=\gamma_3=\gamma_4=:\gamma^1)$$

$$0 \overset{!}{=} PV_{CDS;0,2}(0;R^M_{0,2},L_{\text{gd}};\gamma_5=\gamma_6=\gamma_7=\gamma_8=:\gamma^2)$$

……..

Pricing Credit Default Swaps

constant hazard rate:

$$PV_{CRCDS;0,T} = 0 \text{ if and only if}$$

$$\gamma = \frac{R}{L_{\text{gd}}}$$

Note: the assumption of continuous payment !

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Stochastic Intensity Models

$$r = x + \phi(t;\alpha) \qquad dx = \mu(t,r;\alpha)dt + \sigma(t,r;\alpha)dW$$
$$\lambda = y + \psi(t;\beta) \qquad dy = \mu_\lambda(t,\lambda;\beta)dt + \sigma_\lambda(t,\lambda;\beta)dZ$$
$$dW\,dZ = \rho dt \quad ,$$

W, Z are standard Brownian motions.

constituent short rate models for r and λ

shifted affine term structure models with EBP and EOP

e.g. HW x HW → Schönbucher

CIR++ x CIR++ → Brigo / Mercurio

$$P(0,t) = E[D(0,t)] = E\left[e^{-\int_0^t r(u)du}\right]$$
$$Q(\tau > t) = E[1_{\tau>t}] = E\left[e^{-\int_0^t \lambda(u)du}\right]$$

Independent r and λ

If $\rho = 0$ then both constituent short rate models can be calibrated separately.

1. Calibrate the r-model:

   a) $\phi(t;\alpha)$ as a function of α, from a term structure of $P(0,t)$, which is derived from quoted LIBOR or swap rates.

   b) α according to a term structure of European IR option prices, which corresponds to quoted implied volas.

2. Calibrate the λ-model:

   a) $\psi(t;\beta)$ from a term structure of $Q(\tau > t)$, which is derived from quoted CDS rates.

   b) choose β such that $\psi(t;\beta) > 0$ is sufficiently positive and/or $\int_0^T \psi(t;\beta)^2 dt$ becomes minimal (eventually respecting a predetermined expectation about spread volatilities).

Stochastic Intensity Models

Correlated r and λ

$$\int_{T_0}^{T_n} f(s)E[1_{\tau\in[s,s+ds]}|\mathcal{F}_{T_n}] = \int_a^b f(s)Q[\tau \in [s, s+ds]|\mathcal{F}_{T_n}]$$
$$= \int_{T_0}^{T_n} f(s)\lambda(s)e^{-\int_0^s \lambda(u)du}ds$$

$$PV_{CDS;0,n}(t) = 1_{\tau>t}\Big\{R \sum_{i=1}^{n} \tau_i E[e^{-\int_t^{T_i}(r(s)+\lambda(s))ds}|\mathcal{F}_t]$$
$$+R \int_{T_0}^{T_n} E[\lambda(u)e^{-\int_t^u (r(s)+\lambda(s))ds}|\mathcal{F}_t](u - T_{\beta(u)-1})du$$
$$-L_{\mathrm{gd}} \int_{T_0}^{T_n} E[\lambda(u)e^{-\int_t^u (r(s)+\lambda(s))ds}|\mathcal{F}_t]du\Big\}$$

need: $E[e^{-\int_0^t (r(t)+\lambda(t))dt}|\mathcal{F}_0]$ and $E[\lambda(t)e^{-\int_0^t (r(t)+\lambda(t))dt}|\mathcal{F}_0]$

Stochastic Intensity Models

Gaussian processes for r and λ:

Lemma:
Let x_t and y_t be Gaussian random variables. Then the combined variable

$$A_t \quad := \quad \int_0^t (x_s + y_s) ds$$

is also Gaussian.

Proposition:
Let x_t and y_t be correlated Vasicek processes

$$\begin{aligned} dx &= a(\theta - x(t)) + \sigma dW \\ dy &= a_\lambda(\theta\lambda - y(t))dt + \sigma_\lambda dZ \\ & dW dZ = \rho dt \quad . \end{aligned}$$

$$g(a, T) := (1 - e^{-kT})/a$$

$A = m_A + \sigma_A N_A$ with mean and variance

$$\begin{aligned} m_A &= (\theta + \theta_\lambda)T - [(\theta - x_0)g(a,T) + (\theta_\lambda - y_0)g(a_\lambda, T)] \\ \sigma_A &= (\frac{\sigma}{a})^2(T - 2g(a,T) + g(2a,T)) + (\frac{\sigma_\lambda}{a_\lambda})^2(T - 2g(a_\lambda,T) + g(2a_\lambda,T)) \\ &\quad + \frac{\sigma_\lambda \sigma \rho}{a_\lambda a}(T - g(a_\lambda,T) - g(a,T) + g(a_\lambda + a, T)) \end{aligned}$$

$$\bar{\rho} := corr(N_A, N_y)$$

$$\bar{\rho} \quad = \quad \frac{1}{\sigma_A \sigma_y}\left[\frac{\sigma_\lambda^2}{a_\lambda}(g(a_\lambda, T) - g(2a_\lambda, T)) + \frac{\rho\sigma_\lambda\sigma}{a}(g(a_\lambda, T) - g(a_\lambda + a, T))\right]$$

Stochastic Intensity Models

Lemma:

Let $A = m_A + \sigma_A N_A$ and $B = m_B + \sigma_B N_B$ be Gaussian processes with $\bar{\rho} = corr(N_A, N_B)$. Then

$$E(e^{-A}B) \;=\; m_B e^{-m_A + \frac{1}{2}\sigma_A^2} - \bar{\rho}\sigma_A\sigma_B e^{-m_A + \frac{1-\rho}{2}\sigma_A^2} \quad .$$

this yields finally explicit expressions for the expectations and

$$PV_{CDS;0,n}(0; R, L_{\mathrm{gd}}) \;=\; PV_{CDS;0,n}(0; R, L_{\mathrm{gd}}; \phi(\cdot), \sigma, a; \psi(\cdot), \sigma_\lambda, a_\lambda; \rho)$$

Stochastic Intensity Models

The full G2++ model can be calibrated as follows:

1. Calibrate the HW submodel for r:
   a) determine $\phi(t;\sigma,a)$ as a function of (σ,a), from a term structure of $P(0,t)$, which is derived from quoted LIBOR or swap rates.
   b) calibrate (σ,a) according to a term structure of European IR option prices, which corresponds to quoted implied volas.

2. Calibrate the full correlated G2++-model, keeping $\phi()$, σ, a fixed:
   a) Solve the condition

$$0 \stackrel{!}{=} PV_{CDS;0,n}(0;R^{M}_{0,n}(0),L_{\text{gd}};\phi(\cdot),\sigma,a;\psi(\cdot);\sigma_\lambda,a_\lambda,\rho)$$

   for $\psi(t;\sigma_\lambda,a_\lambda,\rho)$ as function of $(\sigma_\lambda,a_\lambda,\rho)$.
   b) choose $(\sigma_\lambda,a_\lambda,\rho)$ such that $\psi(t;\sigma_\lambda,a_\lambda,\rho)>0$ is sufficiently positive and/or $\int_0^T \psi(t;\sigma_\lambda,a_\lambda,\rho)^2 dt$ becomes minimal (eventually respecting a predetermined expectation about credit spread volatilities and/or the correlation with interest rates).

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CDS Options

A call/put option on a CDS (payer/receiver credit default swaption) gives to its holder the right to enter a CDS (as a protection buyer) at a future time T_0 with a fixed premium rate X. If the option is exercised and default occurs at $\tau \in [T_0, T_n]$, the option holder receives a protection leg (PL) and pays a charge leg (CL) with rate X at payment dates T_i, maximally until default τ.

The payoff of the call/put option on a running CDS with market rate $R^R_{0,n}(T_0)$ is

$$\text{Payoff}_{Call/Put,RCDS;0,n}(t) = D(t,T_0)[\mp \text{Payoff}_{CDS;0,n}(t; X, L_{\text{gd}})]^+$$

$$= D(t,T_0)[\underbrace{\pm \text{Payoff}_{CDS;0,n}(T_0; R^R_{0,n}(T_0), L_{\text{gd}})}_{0} \mp \text{Payoff}_{CDS;0,n}(T_0; X, L_{\text{gd}})]$$

$$= \frac{1_{\tau>t}}{Q\{\tau > t|\mathcal{F}_{T_0}\}} D(t,T_0) \left[Q\{\tau > T_0|\mathcal{F}_{T_0}\} \sum_{i=1}^{n} \tau_i \bar{P}(T_0,T_i) + \right.$$

$$\left. + \text{accrual}_{T_0} \right] [\pm(R^R_{0,n}(T_0) - X)]^+$$

$$\text{accrual}_{T_0} := E[D(T_0,\tau)(\tau - T_{\beta(\tau)-1}) 1_{T_0<\tau<T_n}|\mathcal{F}_{T_0}] \quad .$$

CDS Options

setting

$$\bar{P}^{PR}(T_0, T_i)^{PR} := E_{T_0}[D(T_0, T_i)1_{\tau > T_{i-\iota}}]$$

$\iota := 1$ for PRCDS+

$\iota := 0$ for PRCDS-

the payoff of a PRCDS is

$$\mathrm{Payoff}_{call/put,PRCDS;0,n}(t) = D(t, T_0)[\pm \mathrm{Payoff}_{PRCDS;0,n}(t; X, L_{\mathrm{gd}})]^+$$

$$= \frac{1_{\tau > t}}{Q\{\tau > t | \mathcal{F}_{T_0}\}} D(t, T_0) \left[\sum_{i=1}^{n} \tau_i \bar{P}^{PR}(T_0, T_i) \right] [\pm (R_{0,n}^{PR}(T_0) - X)]^+$$

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CDS Options

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$$PV_{call/put,PRCDS;0,n}(t; R_{0,n}(T_0)) =$$

$$= E[\mathrm{Payoff}_{CDS;0,n}(t; R_{0,n}(T_0), L_{\mathrm{gd}}) - \mathrm{Payoff}_{CDS;0,n}(t; X, L_{\mathrm{gd}})|\mathcal{G}_t]$$

$$= E\Big[\frac{1_{\tau>t}}{Q\{\tau > t|\mathcal{F}_{T_0}\}} D(t, T_0) \underbrace{\sum_{i=1}^{n} \tau_i E_{T_0}[D(T_0, T_i) 1_{\tau > T_{i-\iota}}]}_{=:C^{\iota}_{0,n}(T_0)} [\pm(R_{0,n}(T_0) - X)]^+ |\mathcal{G}_t\Big]$$

$$= \frac{1_{\tau>t}}{Q\{\tau > t|\mathcal{F}_{T_0}\}} E\Big[\frac{1_{\tau>T_0}}{Q\{\tau > t|\mathcal{F}_{T_0}\}} D(t, T_0) C^{\iota}_{0,n}(T_0) [\pm(R_{0,n}(T_0) - X)]^+ |\mathcal{F}_t\Big]$$

$$= \frac{1_{\tau>t}}{Q\{\tau > t|\mathcal{F}_{T_0}\}} E_t\Big[E_{T_0}\Big\{\frac{1_{\tau>T_0}}{Q\{\tau > t|\mathcal{F}_{T_0}\}} D(t, T_0) C^{\iota}_{0,n}(T_0) [\pm(R_{0,n}(T_0) - X)]^+ \Big\}\Big]$$

$$= \frac{1_{\tau>t}}{Q\{\tau > t|\mathcal{F}_{T_0}\}} E\Big[\frac{1}{e^{\int_t^{T_0} r(s)ds}} C^{\iota}_{0,n}(T_0) [\pm(R_{0,n}(T_0) - X)]^+ |\mathcal{F}_t\Big]$$

$$= \frac{1_{\tau>t}}{Q\{\tau > t|\mathcal{F}_{T_0}\}} E^{C^{\iota}_{0,n}}\Big[\frac{C^{\iota}_{0,n}(t)}{C^{\iota}_{0,n}(T_0)} C^{\iota}_{0,n}(T_0) [\pm(R_{0,n}(T_0) - X)]^+ |\mathcal{F}_t\Big]$$

$$= \frac{1_{\tau>t}}{Q\{\tau > t|\mathcal{F}_{T_0}\}} C^{\iota}_{0,n}(t) E^{C^{\iota}_{0,n}}\Big[[\pm(R_{0,n}(T_0) - X)]^+ \Big]$$

$$= 1_{\tau>t} \underbrace{\frac{C^{\iota}_{0,n}(t)}{Q\{\tau > t|\mathcal{F}_{T_0}\}}}_{=:\bar{C}^{\iota}_{0,n}(t)} E^{0,n}\Big[[\pm(R_{0,n}(T_0) - X)]^+ \Big] \quad .$$

CDS Options

Above the numeraire

$$C^{\iota}_{0,n}(t) := \sum_{i=1}^{n} \tau_i E[D(t,T_i)1_{\tau>T_i}|\mathcal{F}_t]$$

is used. In the particular case of an underlying PRCDS−, it holds

$$\begin{aligned}\bar{C}^{0}_{0,n}(t) &= \frac{C^{\iota}_{0,n}(t)}{Q\{\tau > t|\mathcal{F}_t\}} \\ &= \frac{\sum_{i=1}^{n} \tau_i E[D(t,T_i)1_{\tau>T_{i-\iota}}|\mathcal{F}_t]}{Q\{\tau > t|\mathcal{F}_t\}} \\ &= \sum_{i=1}^{n} \tau_i \bar{P}(t,T_i) \quad .\end{aligned}$$

CDS forward rate is martingale w.r.t. the numeraire above

CDS Options

$$PV_{call/put,PRCDS;0,n}(t; R_{0,n}(T_0)) = PV_{call/put,PRCDS;0,n}(t; X, L_{\text{gd}})$$
$$= 1_{\tau>t}\bar{C}^{\iota}_{0,n}(t)\left[N(d_1(t))R_{0,n}(t) - XN(d_2(t))\right]$$

$$d_{1,2} = \frac{1}{\sigma_{0,n}\sqrt{(T_0 - t)}}\left[\ln\frac{R_{0,n}(t)}{X} \pm \frac{1}{2}(T_0 - t)\sigma^2_{0,n}\right]$$

implied volatility $\sigma_{0,n}$ may be
used to quote the price of an CDS option

1. Interest Product Valuation

- Stochastic Interest Models
- Short Rate Models
- Pricing of IR Derivatives
- Market Forward Rate Models

2. Credit Derivatives Valuation

- Stochastic Credit Default Models
- Pricing Credit Default Swaps
- Stochastic Intensity Models
- CDS Options
- **CDS Rate Models**

CDS Rate Models

„CDS spreads“

$$\bar{S}_{0,n}(t) := \frac{R_{0,n}(t)}{L_{\mathrm{gd}}}$$

$$= \frac{\sum_{i=1}^{n} E[D(t,T_i)1_{T_{i-1}<\tau\leq T_i}|\mathcal{F}_t]}{\sum_{i=1}^{n} \tau_i E[D(t,T_i)1_{\tau>T_{i-\iota}}|\mathcal{F}_t]}$$

$$= \frac{\sum_{i=1}^{n} E[D(t,T_i)1_{T_{i-1}<\tau\leq T_i}|\mathcal{F}_t]}{C^{\iota}_{0,n}(t)}$$

for PRCDS-

$$\bar{S}_{0,n}(t) = \frac{\sum_{i=1}^{n} E[D(t,T_i)1_{T_{i-1}<\tau\leq T_i}|\mathcal{F}_t]}{C^{0}_{0,n}(t)}$$

$$= \frac{\sum_{i=1}^{n} E[D(t,T_i)1_{T_{i-1}<\tau\leq T_i}|\mathcal{F}_t]}{\sum_{i=1}^{n} \tau_i \bar{P}(t,T_i)Q\{\tau > t|\mathcal{F}_t\}}$$

structural analogy to the swap rate

CDS Rate Models

periods $[T_{i-1}, T_i]$ of 1 year,

forward CDS rate

$$\begin{aligned}\bar{F}_i(t) &:= \bar{S}_{i-1,i}(t) \\ &= \frac{E[D(t,T_i)1_{T_{i-1}<\tau\leq T_i}|\mathcal{F}_t]}{\tau_i \bar{P}(t,T_i)Q\{\tau > t|\mathcal{F}_t\}}\end{aligned}$$

$\bar{F}_i(t)$ makes the CDS over $[T_{i-1}, T_i]$ a fair contract

Note: Under the condition of independent processes for interest rate and default ($\rho = 0$), it holds (Schönbucher 2000) that (for $\tau > t$)

$$\bar{F}_i(t) = \frac{1}{\tau_i}\left(\frac{\bar{P}(t,T_{i-1})/P(t,T_{i-1})}{\bar{P}(t,T_i)/P(t,T_i)} - 1\right)$$

CDS Rate Models

Normalized CDS rate („spread rate") in terms of CDS forward rates

$$
\begin{aligned}
\bar{S}_{0,n}(t) &= \frac{\sum_{i=1}^{n} \tau_i \bar{P}(t,T_i)\bar{F}_i(t)}{\sum_{i=1}^{n} \tau_i \bar{P}(t,T_i)} \\
&= \sum_{i=1}^{n} \bar{w}_i(t)\bar{F}_i(t) \\
\bar{w}_i(t) &:= \frac{\tau_i \bar{P}(t,T_i)}{\sum_{i=1}^{n} \tau_i \bar{P}(t,T_i)} \quad .
\end{aligned}
$$

Under the condition that the 2-period forward CDS rate $\bar{S}_{i-2,i}(t)$ on $[T_{i-2}, T_i]$ is different from the later (1-period) forward rate $\bar{F}_i(t) = \bar{S}_{i-1,i}(t)$, the following iteration formula holds:

$$
\bar{P}(t,T_i) = \bar{P}(t,T_{i-1}) \frac{\tau_{i-1}(\bar{F}_{i-1}(t) - \bar{S}_{i-2,i}(t))}{\tau_i(\bar{S}_{i-2,i}(t) - \bar{F}_i(t))}
$$